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Dynamics and Stability

by A. Ye. Kobrinskiy

Nauka Press, Moscow, 1964

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By A. Ye. Kobrinskiy

Translation of "Mekhanizmy s Uprugimi Svyazyami. Dinamika i Ustoychivost."
Nauka Press, Moscow, 1964

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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FOREWORD

In modern machines and instruments extensive use is made of elastic /7* elements (springs of different types, membranes, bellows, and the like), which serve as propulsors, converters, pickups, shock absorbers, help force the closure of a mechanism's kinematic linkages, limit maximum internal stresses, and so on. Examples of such devices are mechanisms of instruments with elastic sensing elements, mechanisms of certain electric measuring instruments, clockworks, devices for recording the shape and dimensions of machined articles, weighing devices, mechanisms of conveyer, processing and testing machinery, etc. [15, 58, 77, 79, 90, 110, 112, 117]. This vast class of mechanisms containing elastic elements we are going to call "mechanisms with elastic couplings."

Mechanisms with elastic couplings are frequently designed to function under conditions of vibration or vibration and impact ('vibroimpact'). Such conditions may arise also as a result of the extraneous vibrations and pulsations accompanying the operation of any machine. In either case the problems of design and calculation of mechanisms with elastic couplings are closely tied to analysis of their dynamics and stability.

For that analysis we are often unable to make direct use of the methods of machine or vibration theory. Geometric variability, friction and play in the kinematic pairs, impact interactions of individual parts of the system, are important distinctive features of mechanisms with elastic couplings. Allowing for such features requires further elaboration of those methods, and that is the principal purpose of this book.

The book can be arbitrarily divided into three parts. In the first /8 (Chapters 1, 2 and 3) we formulate the fundamental problems of analyzing the dynamics and stability of mechanisms with elastic couplings, adducing differential equations for their dynamics. Using very simple dynamic models as examples, we describe the stability of the periodic states of motion of vibrating systems and of systems vibrating under impact ('vibroimpact' systems), and in Chapter 1 the basic concepts and definitions are introduced.

Explained also are elements of the theory of differential and difference equations, sufficiently so that the reader will not later have to refer to the many sources that present this subject in a different light and with different notation. Our attention here (in Chapter 2) is directed mainly to solution of equations with periodic coefficients and of difference equations.

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* Numbers in the margin indicate pagination in the foreign text.

This part of the book concludes with a discussion of structural diagrams of mechanisms with elastic couplings, of designs and basic characteristics of the elastic couplings, of vibration exciters, special damping devices, and the damping properties of mechanisms with elastic couplings (Chapter 3).

The next three chapters (4, 5 and 6) constitute the second part of the book, in which we discuss the dynamics and stability of the vibratory states of motion of mechanisms with elastic couplings. We start here by introducing the concepts of the static and frequency characteristics of a mechanism's free vibration, then set up differential equations for its forced vibration, examining the structure of the coefficients of differential equations of motion and introducing the concept of a mechanism's dynamic-equilibrium position as the mean integral value of a generalized coordinate over the period of the external influence (Chapter 4).

The motion of a mechanism with elastic couplings is described by equations with periodic coefficients. Their approximate solution enables us to plot the frequency characteristics and find a mechanism's dynamic-equilibrium position. The difference between the positions of the static and dynamic equilibrium constitute the dynamic error known among instrument manufacturers as "slippage." A simple model is used to explain the physical reasons for it. /9 Explained, too, are the conditions producing resonance and those assuring the stability of vibratory states, and the stability is "charted" (Chapter 5).

A major influence on the dynamics of a mechanism, on its dynamic precision and stability, is exerted by friction in its kinematic pairs. We allow for this influence by introducing a dissipation factor for the mechanism, setting one up for the cases of linear and dry friction and examining the effect of friction on a mechanism's resonance characteristics and stability under different excitation conditions. Again we use simplified models affording a graphic physical picture of a mechanism's motion pattern in the presence of friction in its kinematic pairs (Chapter 6).

The book's third part (Chapters 7, 8 and 9) is devoted to a study of the dynamics and stability of the vibration-under-impact ("vibroimpact") states of motion of mechanisms with elastic couplings.

Starting with a brief survey of machines, devices and systems of the "vibroimpact" type, we adduce dynamic models for them, explaining the method of dynamic analysis of their periodic states of motion. Taking the very simple example of a vibrator "bounding" up a ladder, we illustrate the method for analyzing the stability of periodic states, which is based on the use of finite differences, and chart the stability (Chapter 7).

In the next two chapters two types of "vibroimpact" state of motion are discussed: 1) those wherein elements of the kinematic pairs are subjected to impact on two sides, and 2) those wherein the impact is only on one side.

The dynamics and stability of states of the first type are analyzed with the example of a nonlinear element of the gap or free-play type, for which

solutions are given with and without allowance made for friction, and the stability is charted and test results reported. Then elements of the theory of impact damping of vibration are explained, with a description of the experimental results (Chapter 8).

States wherein the impact is only on one side are discussed in application to a complete two-body elastic system. We determine the free-vibration frequency of a system subjected to periodic impacts, then analyze its forced /10
vibration under the action of a periodic force, also with vibration of the support. We evolve formulae for defining the laws of motion of both parts of the system, their impact velocities, amounts of slippage, and we chart the stability and give the test results. In conclusion we compare the frequency characteristics of the errors due to the system's imbalance, friction and impacts (Chapter 9). Chapters 7, 8 and 9 were co-authored by R.E. Brunshtein, who also wrote the last three sections of Chapter 2.

At the end of the book are two supplements containing: 1) a description of the operation and design principle of the vibration table on which all the tests were run, and 2) mathematical formulae for determining a mechanism's inertial and quasi-elastic coefficients.

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CHAPTER 1

INTRODUCTION

ABSTRACT. Elastic couplings, as contrasted to elastic links, are not actually a part of a mechanism's kinematic linkage. These couplings are analyzed under conditions of vibrating support or pulsating external forces. Vibrating and vibroimpact mechanisms are analyzed. A brief outline of the elements of the theory of differential and difference equations is included. The effects of friction and of clearances on kinematic pairs are analyzed.

§ 1.1. Statement of the Problem. /11

To give a general idea of the scope of the problems dealt with in this book, we must first specify which systems we mean by "mechanisms with elastic couplings." We can best do that by means of very simple diagrams. Figure 1.1.a shows an ordinary six-linkage mechanism, Figure 1.1.b the same mechanisms with elastic couplings k_1, k_2, k_3. As we see, the elastic couplings are not actually part of the mechanism's kinematic linkage. They constitute added devices which include elastic elements and separate interconnecting links of the mechanism.

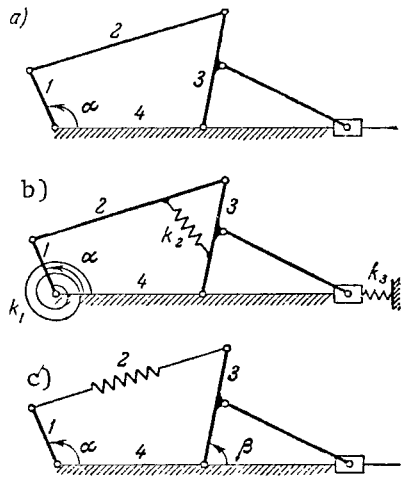

Figure 1.1.

Within the range of movement permitted by the elastic couplings, they impose no additional limitations on the mechanism's mobility and do not affect the number of its degrees of freedom, if the respective masses of the elastic couplings are disregarded.

The features indicated here enable us to /12 distinguish mechanisms with elastic couplings from systems that have come to be known as "mechanisms with elastic links."* Figure 1.1.c shows a mechanism with an elastic link. If we disregard the mass of the elastic link 2, the mechanism has no longer one but two degrees

* See, for instance, S.N. Kuzhevnikov: *Dinamika mashin s uprugimi zven'yami*, "Dynamics of machines with elastic links", Kiev, 1961; also [37].

of freedom, and the position of all its links and points is determined not by one but by two coordinates (e.g., α and β in Figure 1.1.c).

We should note that problems of the dynamics and stability of mechanisms with elastic links are sometimes solved by our assuming that the law of the motion of one of the links is given [e.g., the law of motion $\alpha = \alpha(t)$ of link 1 in Figure 1.1.c]. There the elastic link 2 actually plays the role of an elastic coupling that transmits the exciting force acting upon the mechanism's other links.

The range of movement of the links of a mechanism with elastic couplings is usually defined by several turns, or fractions of a turn, of the driving link, and their movements should be performed at relatively low speeds, the periods of motion alternating with prolonged periods at rest. A large number of the most varied measuring and monitoring instruments, control-system devices, etc., operate in this way. But despite these characteristics, aspects of the dynamics of such systems are of great interest. The reason is that, under the actual conditions under which machines and instruments are used, a mechanism's support or housing may be subjected to various influences such that the true operating picture may be very different from the one designed or calculated on the assumption of the support's immobility and slowness of variation of the applied forces. Simple examples will serve to illustrate. Instruments in airplanes and other conveyances, for instance, are usually fastened to the other structural elements by means of special buffers so that their precision and durability will not be adversely affected by the vibration from the engines and the action of external forces. But despite the /13 buffers, some of the vibration does get transmitted to the instrument housings through the links of the mechanisms immovably connected to them. Experience has shown that these vibrations importantly affect an instrument's behavior and accuracy [32, 58, 87].

Pneumatic measuring mechanisms are another example. They are fed compressed air, whose pressure during operation must remain constant. Owing to the interaction of various factors, however, the pressure of the air does not remain constant but "pulsates" about a certain mean value (see [104], for instance). Then the system's operating state is no longer static but dynamic, whence only a dynamic analysis can portray its performance and operating precision.

The above examples afford some idea of the range of problems associated with analyzing mechanisms with elastic couplings functioning under conditions of a vibrating support or pulsating external forces.

The vibration and pulsation are regarded here as extraneous influences inevitably accompanying the operation of any machine. As machines are driven faster and under greater and greater loads, these influences become ever more significant. It is important that ways be found to eliminate their effects on the functioning of a machine's individual elements and mechanisms, and on the reliability and precision of the equipment for servicing them.

If in one broad class of mechanisms with elastic couplings vibrational states arise as a result of unavoidable extraneous influences, in another, no less broad class of machines and instruments such states are the very basis of the job or task that the mechanisms are designed to perform.

Vibration-measuring instruments, vibrating conveyers, vibratory diggers, vibration dampers, vibrosealants, and numerous other vibrating machines, instruments and devices afford a graphic illustration of the effective utilization of vibrational states of motion.

The two classes of system, as we see, have very different functional purposes. But when we speak of the vibrational stability or vibration-proofness of mechanisms with elastic couplings functioning under conditions of extraneous vibration effects, it emerges that in many cases the problems that have to be solved are similar to those encountered in the designing of mechanisms with elastic couplings directly intended to operate in a vibrating state. In the case of both classes, the dynamic-analysis results can be used to select a mechanism's parameters and design so as either to minimize undesired vibration effects or, on the contrary, maximize the desired vibrational states. /14

In the course of the last few years, along with the study of vibrational states, there has been considerable research done on the vibration-under-impact ("vibroimpact") states of motion of mechanisms with elastic couplings. The need to study "vibroimpact" states arises primarily in connection with the calculation of mechanisms and systems designed to operate under conditions of vibration and impact. By now such systems encompass a vast group of different machines and devices, among which: vibratory cutting and breaking tools, vibratory impact diggers, machines for vibrational impact testing, impact-type vibration dampers, etc.

Interactions between vibration and impact may also occur as an unplanned and at times very disagreeable phenomenon accompanying a system's normal operation. The characteristic rattling sometimes observed, for instance, in instruments with elastic elements operating under conditions of vibration indicates that in the moving joints of these instruments there are vibro-impact interactions affecting their individual links and parts.

The clearances and play in the mechanisms of machines, instruments and mechanical linkages of control systems create conditions favoring the occurrence of such phenomena whenever pulsating forces and moments act upon the system's elements.

Development of effective methods in some cases, for eliminating vibro-impact interactions and, in other cases, for choosing system parameters that will maximize vibroimpact states requires study of the dynamics of those systems and determination of the reaction of a mechanism with elastic couplings to an external influence applied to it. /15

The reaction of a mechanism to a given stimulation or excitation, like

the reaction of any mechanical system, depends not only on the stimulation but on the mechanism's parameters. By its parameters we mean its structure, geometric dimensions, configuration, the magnitudes and character of the distribution of the masses and moments of inertia of its individual links and their elasticity, the characteristics of the elastic couplings and their masses.

Of great importance are the parameters for the actual design of a mechanism's kinematic pairs, and especially for the friction between its elements and for the additional relative mobility of those elements due to the clearances.

The large number of jointly operating factors immediately raises the question as to the possible methods of analyzing their influence on the mechanism's behavior. Obviously, there are only two methods. The first -- precise solution of the problem through simultaneous allowance for all the indicated parameters --- would result in equations so vast that it would be very difficult to apply them in most cases.

The second (approximative) method is based on taking into account only one or a few parameters whose influence is seen to be predominant from examination of the mechanism's design and operating conditions. This method is used especially in mechanism-precision theory, where its application rests on the principle of the independence of the effect of primary errors in the event that they are small. This principle greatly simplifies the problem of analyzing the geometric and functional precision of mechanisms (see, for example [15, 17, 18, 41 and 80]). For solving dynamic problems this method is often inapplicable owing to its fundamental lack of any formal method for combining the results of analyses wherein different groups of parameters were taken into account.

Even so, this is the method we use, because for solving a number of /16
important problems it permits us to obtain clear and understandable results. Both qualitatively and quantitatively the results are sufficiently precise when preliminary analysis of the mechanism enables us with some confidence to indicate the parameters exerting maximum influence on the phenomonon under study, e.g., on the magnitude of the mechanism's dynamic error or on establishment of an intense vibroimpact state of motion.

In other cases these results give us only qualitative estimates, which do allow us, though, in designing a mechanism to select its basic parameters more knowingly and, above all, to gain a clear conception of their physical effect on the behavior of the system as a whole.

Of the basic mechanism parameters enumerated above, we are disregarding two of them, the elasticity of the links and the mass of the elastic couplings. In other words, we are assuming that the compliances of the elastic couplings are incommensurably great, their mass incommensurably small, compared to the compliances and masses of the mechanism links.

§ 1.2. Vibration and Pulsation.

In studying the influence of a vibrating support or pulsating forces on the dynamic precision and stability of mechanisms with elastic couplings we have to keep in mind two typical features of these excitation sources.

In an overwhelming majority of cases vibration or pulsation occurs as a result of the joint action of a large number of the most varied factors. To determine the specific effect of any one of those factors on the process as a whole is difficult. Actually, returning to our examples, we need only remember that the amplitude and frequency of the vibrations of the bodies or hulls of transport machines depend not only on the dynamic properties of the shock absorbers but on the frequency of the free vibrations of the structural elements to which the bodies or hulls are fastened, on the operating conditions of the engines, on the nature of the machine's motion, etc.

The amplitude and frequency of the air-pressure pulsation in the system of a pneumatic measuring mechanism depends on the number of the compressor's revolutions, ratio of the receiver volume to the compressor's working volume, length of the conduit and, finally, on the operating conditions of the pneumatic system as a whole -- in other words, on the evenness of the air consumption [sic] as the resistances at the outlet vary, etc. The vibration of land machines, installations and apparatuses results from the operation of engines, electric motors, pumps, compressors, etc. /17

It is obvious that theoretical determination of even such special features of the vibration and pulsation processes as their frequency and amplitude poses very great difficulties. As for plotting complete characteristic curves for these processes, i.e., of finding, for example, the position of an instrument or of a pressure in a pneumatic system as a function of the time, any attempt to do this by the usual analytic methods is foredoomed to failure owing to the large number of initial parameters and impossibility of determining with sufficient accuracy the nature of their influence on the process as a whole.

These difficulties necessitate our resorting to a systematic experimental investigation and subsequent statistical analysis of the results. For that purpose we make an experimental study of the steady-state operation of a group of selected machines, picking out of the complex pattern of their vibration or pulsation the "main" components of those processes, whose frequency and amplitude may materially affect the behavior of a system subjected to their influence [78, 87]. Knowing the possible frequency and amplitude spectra will enable us to design effective protective devices, draw up technical specifications for the redesigned mechanisms and instruments, and program their testing with special vibration- and vibroimpact-test equipment.

In Table 1.1 below, frequency and amplitude data on certain transport machines are given as an example.

TABLE 1.1 AMPLITUDES AND FREQUENCIES FOR CERTAIN TRANSPORT MACHINES

Type of machine*	Amplitude (mm)		Frequency (Hz)	
	From	To	From	To
Airplane, center section	0.025	0.40	10	500
Airplane, nose and tail	0.025	0.50	10	500
Warship, large	0.75	2.5	15	25
Patrol or PT boat	0.025	2.5	-	50
Ground vehicles, wheeled	0.1	5.0	15	120
Ground vehicles, track-type	0.025	5.0	10	500

* In this table the maximum vibration amplitudes correspond to their minimum frequencies.

So the first characteristic of the excitation processes whose effect upon mechanisms with elastic couplings we are considering is that in most cases the exact laws governing them cannot be determined theoretically. As a rule, we /18
know, from testing, only the limits of the possible amplitudes and frequencies of their "main" components.

Taking this into account in what follows, when we come to discuss the dynamics and stability of systems operating under conditions of vibration or pulsation, we shall confine ourselves to cases of steady states of excitation. We shall consider that the excitation is a process whose frequency and amplitude are random variables which remain constant during the period of time under study. This being so, it becomes entirely admissible to replace a certain essentially random function of the excitation with a periodic function of the random variables.

As already mentioned, vibration and pulsation constitute extraneous, usually harmful influences. The ultimate aim of all theoretical and experimental studies of them is to eliminate or minimize their influence on a system.

We know that the proper functioning of an airplane instrument depends largely on how well it is protected from vibration, whereby we mainly mean /19
the amplitude of the vibrations of the instrument housing.

Naturally, the efforts of the designer are always aimed at minimizing that amplitude. In exactly the same way, a pneumatic instrument, the pressure in whose measuring system pulsates, operates reliably and with sufficient accuracy only when the pulsation amplitude can be kept relatively small, which is assumed by various design features.

Because in the actual designing of mechanisms, machines and instruments

special measures are taken to eliminate all possible extraneous influences, the amplitudes of such processes can be considered small.

The "smallness" of the excitation amplitude is relative, and must therefore be allowed for in the practical problems that we shall be discussing. Henceforward we shall consider the vibration amplitude small if the absolute displacements of a mechanism's support are small compared to the absolute dimensions of its links. We shall regard a pulsation amplitude as small if the mechanism's deviation from the equilibrium position, caused by the static action of a force equal in magnitude to the pulsation amplitude, is small compared to the mechanism's absolute dimensions.

Our assuming a relative smallness of the excitation amplitude still does not tell us what the results will be of its influence upon the mechanism's operation. In fact, high-frequency vibrations of the knives of metal-cutting lathes, whose amplitude rarely exceeds some tenths of a millimeter, result in serious disturbances of the machining process [35, 67]. Vibration amplitudes of instrument housings, which seldom exceed the order of 1-2 mm, may be the source of substantial dynamic errors, etc. Even for a crude estimate of the intensity of excitation any assumptions about its amplitude are still inadequate. A second, no less important factor in that estimate is the excitation frequency. Along with that, our assuming a relative smallness of the excitation amplitude in a way simplifies solution of the problems posed, permitting in some cases linearization of the equations describing the motion /20 of the mechanism throughout a complete vibration period or for the separate intervals thereof.

We have described briefly the characteristics of the excitation to which mechanisms with elastic couplings may be subject in actual use, and have indicated how those characteristics are reflected under the conditions that we assumed. In what follows, we shall not concern ourselves with analyzing the causes of vibration and pulsation. Their character and range of possible frequencies will be considered given, and their amplitudes will be assumed small in the sense defined above.

In vibratory and vibroimpact-type machines and instruments special devices known as vibrators are used as the source of the pulsating force. The dynamic parameters of vibrators, the frequencies and amplitudes of the forces that they develop, are determined by the designer as he designs the mechanism. So here too we can consider all the excitation parameters given.

We have thus indicated the range of problems that will be solved in this book, and the assumptions that we have adopted. Now we are going to show, by means of very simple examples, the mathematical equations to which we are led by the methods used to solve those problems.

§ 1.3. Dynamic Models.

An example of a very simple two-link mechanism is a compound pendulum. It is a solid body capable of rotation about a horizontal axis (Figure 1.2.a).

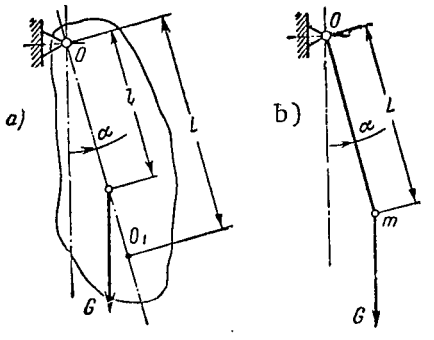

Figure 1.2.

Displaced from its equilibrium position, it sways about the axis of its point of support O.

The pendulum's position is defined by the angle α measured from the vertical line coinciding with its static-equilibrium position. The equation for the swaying motion is obtained by using Newton's second law, according to which

$$J\frac{d^2\alpha}{dt^2} = -Gl\sin\alpha. \qquad (1.1)$$

On the right-hand side of this equation we have the moment of gravity, which in the swinging of a pendulum is directed in the sense opposite to that of the growth of the angle α, hence in equation (1.1) appears with a minus sign. /21

Henceforward derivatives of a function of the time will frequently be designated with dots over the letters. The pendulum's angular velocity and acceleration are thus designated as

$$\frac{d\alpha}{dt} = \dot{\alpha}, \qquad \frac{d^2\alpha}{dt^2} = \ddot{\alpha}.$$

The pendulum's moment of inertia we express in terms of its mass and radius of inertia: $J = mr^2$. We then get

$$\ddot{\alpha} + \frac{gl}{r^2}\sin\alpha = 0, \qquad (1.2)$$

where g is the acceleration of free fall. If the pendulum's oscillation range is not great, we can approximately substitute

$$\sin\alpha \approx \alpha,$$

where α is expressed in radians. Thus, for example, if $\alpha_0 = 0.25$ (approximately 15°), then with that substitution the error in the magnitude of the moment of gravity will not exceed 1%. Here α_0 is the pendulum's maximum departure from its equilibrium position or oscillation amplitude. /22

The equation for the pendulum's motion now assumes the following form:

$$\ddot{\alpha} + \omega_n^2\alpha = 0. \qquad \omega_n^2 = \frac{gl}{r^2}. \qquad (1.3)$$

Later we shall see that this same equation describes the small free oscillations of a mechanism with elastic couplings that possesses one degree of mobility. So a pendulum mechanism can be considered a dynamic model of a mechanism with elastic couplings.

The problem of dynamic analysis of our model consists in finding the value of α as a function of the time t. For this we must seek solutions of equations (1.2) or (1.3) relating the sought-for function α to its time derivatives.

Equations of the (1.3) type, in which the function α and its derivatives come in in the first degree, are called linear. The operation whereby we approximatively replaced the nonlinear function in equation (1.2) with a linear one is called linearizing the equation.

The reader can readily verify that the function $\alpha = \alpha_0 \sin \omega_n t$ satisfies the linearized equation (1.3), hence is its solution.

So the small oscillations of a compound pendulum closely approximate harmonic motion. The basic characteristic of that motion, along with the amplitude α_0, is its frequency or oscillation period T.

The period of the small oscillations of a pendulum is

$$T = \frac{2\pi}{\omega_n} = \frac{1}{\nu},$$

where ν is the oscillation frequency, ω_n the angular frequency.

Let us note, incidentally, that the period of the small oscillations of a compound pendulum is exactly equal to the period of the small oscillations of /23
a so-called simple pendulum, which constitutes a mass point equivalent to the mass of a compound pendulum suspended on a weightless thread or rod (Figure 1.2,b) of length

$$L = \frac{r^2}{l}.$$

The point O_1 (Figure 1.2,a) of the compound pendulum is called the center of oscillation or center of percussion. We know that the point of support and center of oscillation are interchangeable, in other words, that the periods of the small oscillations of a pendulum about an axis passing through any of these points are identical.

The differential equation (1.2), in which the sought-for function $\alpha = \alpha(t)$ comes in under the sine symbol, is nonlinear. Solving it is not so simple as solving a linear equation. The reason is that there are still no rigorous mathematical methods for precise solution of nonlinear differential equations, though there do exist many ways of finding approximate solutions

for them. Hence in solving engineering problems, which essentially are all nonlinear, we try to confine ourselves to consideration of linearized systems in order to use the well-developed methods for solution and analysis of linear equations. In short, for an approximative study of an exact (nonlinear) system we endeavor to substitute a precise study of some approximate (linear) system. So we too, of course, wherever possible, shall proceed in the same way.

And now we shall take a look at the equations for the dynamics of mechanisms with elastic couplings operating under conditions of a vibrating support or pulsating external force. We shall assume that the point of support or a simple pendulum does not remain stationary but oscillates as $x = x(t)$ along a certain straight line x - x forming an angle β with the vertical axis (Figure 1.3). This point of support may be connected, for instance, to the housing of an instrument subject to vibration. Understandably, the main job of dynamic analysis here becomes solution of the problem as to how the mechanism moves with respect to the vibrating housing. So now we have to put together an equation for the motion of a pendulum with respect to a certain moving system to which its point of support is connected. /24

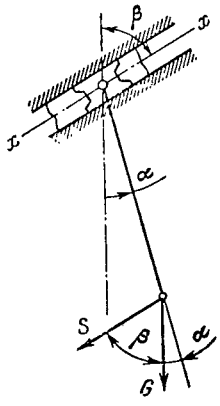

Figure 1.3.

From theoretical mechanics we know that the differential equations for the dynamics of such relative motion are set up in the same way as the equation for motion with respect to a fixed point, except that, to the forces directly applied, we have to add an inertial force dependent on the motion of the relative system. Since we assumed that the point of support moves in a straight line, we must write the inertial force S of the transportation as

$$S = -m\ddot{x}(t).$$

Now regarding S as the usual external force applied to a pendulum, we can put together an equation for its relative motion by bringing in, as before, Newton's second law:

$$mL^2\ddot{\alpha} = -mgL\sin\alpha - m\ddot{x}(t)L\sin(\alpha+\beta),$$

whence

$$\ddot{\alpha} + \frac{g}{L}\sin\alpha = -\frac{1}{L}\ddot{x}(t)\sin(\alpha+\beta). \tag{1.4}$$

Please note that this is roughly the form of the equation for the motion of a pendulum in the case wherein a mass m is acted upon directly by a certain pulsating force $p = p(t)$, e.g., one generated by a directional vibrator whose housing is connected to that mass.

If we confine ourselves to considering only the pendulum's small oscillations, approximatively substituting cos $\alpha = 1$ for sin $\alpha \approx \alpha$, as a result of that linearization we get instead of (1.4) the equation:

$$\ddot{\alpha} + \frac{g}{L}\alpha = -\frac{\ddot{x}(t)}{L}(\alpha\cos\beta + \sin\beta).$$

We rewrite it in the form /25

$$\ddot{\alpha} + \frac{1}{L}[g + \ddot{x}(t)\cos\beta]\alpha = -\frac{\ddot{x}(t)}{L}\sin\beta. \tag{1.5}$$

Equation (1.5), like (1.3), is a linear differential equation. But there are important differences between them. First of all, the coefficient with the function α in equation (1.5) is a function of the time. Next, its right-hand side does not equal zero. It is a nonhomogeneous linear equation with a variable coefficient.

If the vibration of the support or external-force pulsation has a periodic character so that

$$\ddot{x}(t + T) = \ddot{x}(t),$$

or

$$p(t + T) = p(t),$$

T being the vibration or pulsation period, then equation (1.5) will have a periodic coefficient. An equation of this type is called a nonhomogeneous Hill equation.

If the vibration of the point of support obeys the harmonic law $x = x_0 \cos \omega t$, equation (1.5) assumes the form:

$$\ddot{\alpha} + \frac{1}{L}(g - x_0\omega^2\cos\beta\cos\omega t)\alpha = \frac{x_0\omega^2}{L}\sin\beta\cos\omega t. \tag{1.6}$$

This is called a nonhomogeneous Mathieu equation. In the special case wherein the vibration direction coincides with the direction characterizing the pendulum's static-equilibrium position ($\beta = 0$ or $\beta = \pi$) we get the homogeneous Mathieu equation

$$\ddot{\alpha} + \frac{1}{L}(g - x_0\omega^2\cos\omega t)\alpha = 0. \tag{1.7}$$

Finally, if the direction of vibration is characterized by the angle $\beta = \pi/2$, the differential equation for the pendulum's motion will have the form:

$$\ddot{\alpha} + \omega_n^2 \alpha = \frac{x_0 \omega^2}{L} \cos \omega t. \tag{1.8}$$

This is a nonhomogeneous linear equation with constant coefficients. It /26 differs from equation (1.3) in having a periodic function of the time on its right-hand side.

We shall be encountering equations of the (1.6), (1.7) and (1.8) type throughout this book. It will be useful, therefore, to have a more detailed knowledge of the elements of the theory of such equations. The next chapter will be devoted to them.

§ 1.4. Models of Vibroimpact Mechanisms.

As our model of a vibroimpact mechanism we again take a simple pendulum. Only now its motion is limited by an immovable wall (Figure 1.4) against which it strikes.

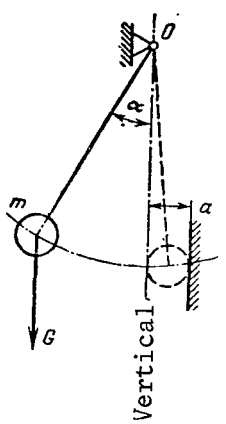

Figure 1.4.

What we call an impact is a kind of motion wherein, in the space of a very short period of time (e.g., thousandths or ten-thousandths of a second), a system's velocity undergoes finite but not small variations. As the result of an impact, the system's momentum, too, undergoes a corresponding finite variation.

Applied to our problem, this means that a pendulum, having struck against a wall with a velocity $L\dot{\alpha}_1$, at the instant t_1 a short time τ afterward (time of impact) will rebound at a different velocity $L\dot{\alpha}_2$, the change in the velocity having a finite value, through the period of impact was extremely short. The momentum, by which in this case we mean the product of the moment of inertia multiplied by its angular velocity, will then vary by

$$\Delta Q = Q_1 - Q_2 = mL^2 (\dot{\alpha}_1 + \dot{\alpha}_2).$$

The forces that develop in the process of an impact reach extraordinarily high values, and it is only the brevity of their action that limits their effect on colliding bodies. To stress this aspect of them, forces of collision/27 of impact are called instantaneous forces. All other forces applied to a system during a period of impact are unable to exert upon it any appreciable influence. So, in studying the impact phenomenon, we can completely disregard all non-instantaneous forces.

We have already seen that in the course of an impact the velocity of the colliding bodies changes abruptly, yet during that time they are able to undergo no significant displacement, hence for calculation purposes can be considered stationary.

To calculate the impact of real bodies in impact theory, experimental coefficients are used, which have been determined hitherto on the basis of a hypothesis introduced by Newton, according to which the ratio of a body's velocities after and before impact v_2/v_1 is a constant, dependent neither on the impact velocity nor on the dimensions of the colliding bodies but only on their on the nature of their material, i.e.,

$$|v_2| \quad : \quad |v_1| \quad = R.$$

The positive quantity R is called the coefficient of restitution. It characterizes the extent to which a body's velocity is restored after an impact. This coefficient can also be given a simple physical interpretation. For that we break the impact period down into two subperiods: 1) the subperiod τ_1 from the instant of initial contact to that at which the colliding bodies are closest together, during which time the colliding surfaces undergo maximum deformation; 2) the subperiod τ_2 from the instant at which the colliding bodies are closest together to that at which the colliding surfaces separate, during which time the undeformed state is restored.

If the colliding bodies are absolutely elastic, their deformation will disappear completely after the impact, and the impact process will proceed symmetrically during the subperiods τ_1 and τ_2, which in this event will be equal. Accordingly, the relative velocity of the bodies will be fully restored, and the coefficient of restitution will be R = 1.

If the impact is absolutely inelastic, there will be no subperiod τ_2 at all, and the impact will end with the maximum deformation of the colliding surfaces, without subsequent restoration of the original configuration. Then the relative velocity of the bodies after the impact will equal zero, hence R = 0. In the case of real physical bodies the coefficient of restitution is always $0 \leq R < 1$. /28

In mechanics handbooks and other reference works the following values of R are given for:

Wooden balls	0.50
Steel balls	0.55
Ivory balls	0.89
Glass balls	0.94

But in analyzing and calculating vibroimpact systems we frequently have to resort to idealization of the impact process, assuming a coefficient of

restitution R = 1 or R = 0.

The impact phenomenon greatly complicates the job of analyzing the dynamics of systems even in cases in which, without going into the essence of the phenomenon, we allow for its effect only to the extent of the restitution coefficient. The trouble is that vibroimpact systems are basically nonlinear.

Figure 1.5 shows the characteristic of the so-called restoring moment for an ordinary simple pendulum performing small oscillations. The farther we remove the pendulum from its equilibrium position, the greater is the moment necessary for that. When the direction of the deviation changes, the sign of the moment changes. The magnitude of the moment is a linear function of the pendulum's deviation angle. This moment figures in differential equation (1.3) for a pendulum's small oscillations. Because it is linear with respect to the sought-for function α, the differential equation itself is linear.

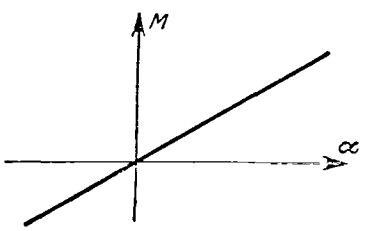

Figure 1.5.

But in the case of a pendulum with a stop, the restoring moment characteristic is quite different, as shown in Figure 1.6, where the stop is assumed to be at a distance a from the pendulum's static-equilibrium position.

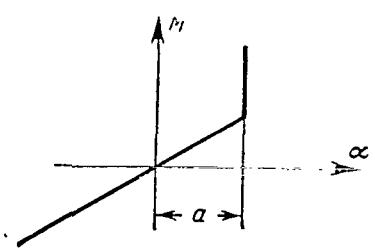

Figure 1.6.

As we see, the linear interdependence of the moment and of the deviation of the system that it causes holds only up to the value a characterizing the impact coordinate. Then the moment's characteristic breaks, the linear interdependence is gone, and the linear equation for the motion ceases to apply. And no matter how much we decrease the amplitude of a vibroimpact system's oscillations, the system cannot be linearized. That is why vibroimpact systems are referred to as essentially nonlinear. /29

We note, though, that in the intervals between impacts the pendulum's motion does continue to obey the linear equation. Henceforward we shall make extensive use of this fact by "fitting" the individual motion intervals together so as to allow for the variations that make up the motion of a system of which separate parts collide.

We assume the pendulum and the stop (Figure 1.4) to be made of absolutely elastic materials (R = 1). We move the pendulum away from its equilibrium position and let it go. Then, obeying equation (1.3), it will move harmonically right up to the instant of striking the stop. Since R = 1, the pendulum's velocity at the instant after impact will exactly equal its velocity at the instant before impact.

As a result, the pendulum will swing, again striking the stop, etc. Its motion will be strictly periodic, as shown by the solid line in Figure 1.7. The broken line in that figure graphs the motion of the pendulum in the linear case. We readily see that the periods of the motion without impact (T) and with the impacts (T*), and the corresponding/30 motion laws, are quite different.

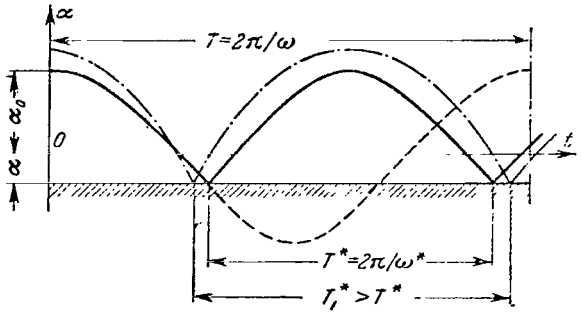

Figure 1.7.

As we know, the period of free linear oscillations does not depend on the magnitude of the initial deviation. With the free oscillations of a pendulum under vibroimpact conditions, the oscillation period T* does vary as a function of the initial deviation (see Figure 1.7, where $T^*_1 > T^*$). This is one of the properties of nonlinear systems that we are going to have to face in our study of vibroimpact mechanisms.

A pendulum with a stop, like a free pendulum, constitutes a system with one degree of freedom. But in studying vibroimpact systems we shall not be able to confine ourselves to only those with one degree of freedom. Whether allowing for clearances in kinematic pairs, designing vibroimpact-type machines, or calculating the impact damping of vibrations, in all these cases we have to analyze the motion not of one-body but of two-body systems.

The double pendulum shown in Figure 1.8 may serve as a model of such a system. The first thing about it that we look at is its free oscillations. If we ascribe to it conservative properties, i.e., assume that both pendulums are absolutely elastic, and if we disregard the losses due to friction at the point of support and the resistance of the surrounding medium, the system's free oscillations will be periodic in nature (Figure 1.9). In the course of those oscillations the instants of impact /31 will alternate with intervals of motion during which equation (1.3) will hold for each of the system's parts.

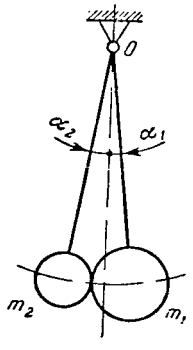

Figure 1.8.

In addition to the two-body system's free-oscillation period T*, we shall be determining the impact coordinate α_c. In the general case, you see, it need not coincide with the coordinate characterizing the location of the system's static-equilibrium position.

Finally, we shall have to investigate the forced oscillation of two-body vibroimpact systems due to external excitation. As already mentioned, we shall confine ourselves to cases wherein the excitation is steady and periodic. Under the influence of a periodic

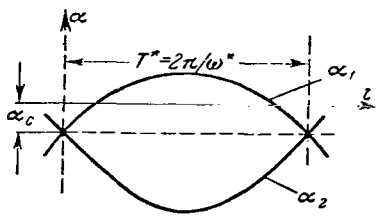

Figure 1.9.

excitation a vibroimpact system may perform periodic motions. Our problem will be to ascertain the corresponding periodic states, with equations (1.5)-(1.7), and the conditions, flowing from the momentum theorem, for "fitting together" the adjacent intervals of motion.

We recall that, according to this theorem, in the absence of external instantaneous forces the system's momentum before and after impact stays the same. To simplify, we write this theorem for the case wherein both masses m_1 and m_2 move forward along a certain axis X. If their velocities prior to impact were u_1 and v_1, respectively, and u_2 and v_2 after impact, then according to the momentum theorem /32

$$m_1u_1 + m_2v_1 = m_1u_2 + m_2v_2. \tag{1.9}$$

But this equation is not enough for us, knowing the velocities of the bodies prior to impact, to determine how they will vary as a result of the impact. For that, we still have to stipulate the nature of the impact, i.e., the restitution coefficient. Since now both the bodies are moving, by restitution coefficient we mean the quotient of the relative rate of rebound divided by the relative rate of fall, i.e.,

$$R=\frac{v_2-u_2}{u_1-v_1}. \tag{1.10}$$

Here the signs are selected so that R is greater than zero. From equations (1.9) and (1.10) we find

$$\begin{aligned} u_2&=\frac{(m_1-Rm_2)\,u_1+m_2(1+R)\,v_1}{m_1+m_2},\\ v_2&=\frac{m_1(1+R)\,u_1+(m_2-Rm_1)\,v_1}{m_1+m_2}. \end{aligned} \tag{1.11}$$

These two equations enable us to link the conditions of the motion of both the system's parts at the end of one motion interval to the conditions of their motion at the start of the next.

§ 1.5. The Concept of Dynamic Stability.

We recall that finding a system's static stability involves determining its equilibrium conditions and ascertaining what happens to it when, as a result of some small perturbation, it is displaced from its equilibrium position. A perturbation imparted to a system in a position of unstable equilibrium will cause it ever increasingly to depart from that position.

When a system is in a position of stable equilibrium, if displaced from it, it will tend to return to it.

A pendulum (see Figure 1.2) has two equilibrium positions ($\alpha = 0°$, $\alpha = 180°$). The problem as to which of these positions is stable, in other words, the problem of the pendulum's static stability, is easily solved by using the stability criteria and the indicated method for bringing in the perturbations. /33

The problem of a system's dynamic stability sets up in about the same way. Only then we are talking about the stability of one or another of the system's motions rather than one or another of its equilibrium positions. Let us assume that the pendulum's point of support is vibrating harmonically in a vertical direction. We already know that the pendulum's motion then obeys a homogeneous Mathieu equation, which we write as:

$$\ddot{\alpha} + \left(\omega_n^2 - \frac{x_0\omega^2}{L}\cos\omega t\right)\alpha = 0, \qquad (1.12)$$

where ω_n is the angular frequency of the pendulum's free oscillations, ω the angular frequency of its vibration.

If the pendulum is in a position of statically stable equilibrium, it may seem at first glance that its moving along with its point of support will not affect the stability of the position that it occupies with respect to the vibrating support.

Actually, though, this is not always the way things are. With certain ratios between the frequencies ω and ω_n a statically stable position of the pendulum is dynamically unstable. Then, as a result of a very minor initial perturbation, the pendulum will start to swing with greater and greater intensity. To study the pattern of this "buildup," which illustrates the dynamic-instability phenomenon, we take as example a case wherein the excitation frequency $\omega = 2\omega_0$, i.e., is twice as great as the pendulum's free-oscillation frequency, and we replace vibration of the support with pulsation of the external force $P = P_0 \cos\omega t$. In the two cases the equation for the motion has the same form, since the pattern of the pendulum's motion will be the same in both.

We let the pendulum receive a certain initial perturbation, e.g., a random bump, shake, twist of its support axis, or any other such accident likely to befall the operation of a real system. For us the only thing that matters is that, as a result of this perturbation, the pendulum has started to swing.

Noting on the diagram of the pendulum's swinging motion (Figure 1.10) the points a and c, which characterize its extreme positions, we add to that figure the diagram of a pulsating force. Depending on the condition, the period of this force is two times shorter than that of the pendulum's free oscillations. /34

We observe how the moment of this force is directed during the different intervals of the pendulum's motion. In the interval 0 - a the pendulum moves out to one of its extreme positions. During that time the force P acts upward, so that the direction of the force's moment coincides with the direction of the angular velocity of the pendulum's swinging, whence during this interval the pulsating force does positive work. We have an analogous pattern during the second half-period of the pendulum's oscillation. As it moves from the equilibrium position to point c, the force will be directed upward, but downward as it moves to the equilibrium position. So, throughout the pendulum's entire oscillation period the pulsating force will be performing positive work; the system's energy will grow; the amplitude of the pendulum's swinging will increase. The pendulum's statically stable position is dynamically unstable.

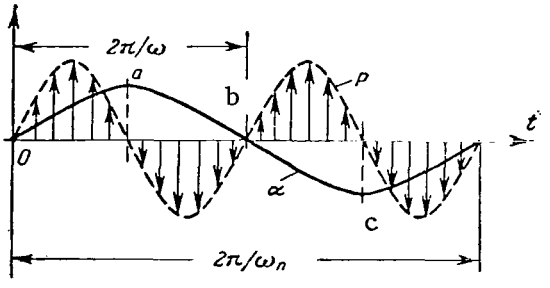

Figure 1.10.

The period of the free oscillations of a linear system does not depend on the oscillation amplitude. Hence increasing the linear-oscillation amplitude will not affect the nature of the influence of a pulsating force on a pendulum. The described buildup pattern may continue for any length of time, during all of which the pendulum's oscillation amplitude will continuously increase. /35

It goes without saying that such a result, which over an infinitely long time leads to an infinitely large oscillation amplitude, flows from formal analysis of an idealized, exactly linear system. Real physical and engineering systems are always to some extent nonlinear. For a pendulum's large oscillation amplitudes our assumed linearization turns out to be too crude. It is too crude in the sense that in reality a pendulum's oscillation period does increase as its oscillation amplitude increases.

With large amplitudes the increase of the oscillation period becomes substantial, and the system's nonlinearity starts to make itself felt. As a result, our above depiction of the buildup pattern grows less and less applicable. It becomes clear that, when the nonlinear factors are taken into account, the system's oscillation amplitude will not increase to infinity.

But this should give us no comfort. Very intense vibration, errors in instrument readings, heavy dynamic loads, fatigue failures are just some of the unpleasant results of a system's operating under such unstable conditions. The very fact that a system's oscillations show a tendency toward unlimited increase is enough in itself to make us start thinking about changing either its design or operating conditions.

We have examined the case wherein the frequency of the pulsating force is twice as great as the free-oscillation frequency, and found the the corresponding state of motion is unstable. But the swinging of a pendulum,

in other words, the unstable states of its motion, are also possible with other ratios between the frequencies ω and ω_n. Mathematical analysis will enable us later to determine the entire ranges of the stable and unstable states of motion of mechanisms with elastic couplings and thereby greatly supplement our dynamic-analysis results. /36

We have now shown how, as a result of dynamic influences, a pendulum's statically stable position becomes unstable. We know that the reverse is also possible. In Chapter 5 we shall show that a statically unstable position of a pendulum ($\alpha = 180°$) can be stable when the point of support vibrates. In this case the dynamic influence exerts a stabilizing effect on the system.

From this we may conclude that a mechanism's positions of static and dynamic equilibrium can be very different. This premise will have great importance in what follows.

§ 1.6. Stability of the Motion of Vibroimpact Systems.

To get an idea of the substance of the problem of analyzing the stability of the motion of vibroimpact mechanisms and systems, we take a look at another dynamic model (Figure 1.11). It is a platform 1 that moves harmonically in a vertical direction. A little ball 2 made of an elastic material (e.g., steel) is thrown down onto it. Obviously, when the platform's oscillations are sufficiently intense, the little ball will start to bounce, striking the platform and rebounding from it.*

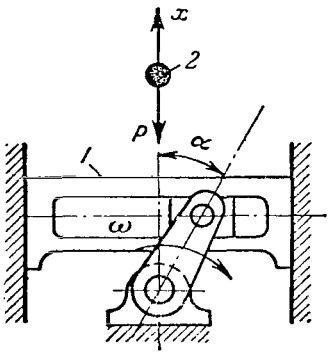

Figure 1.11.

We shall first determine under what conditions this motion of the ball will be periodic in nature, i.e., under what conditions the ball will bounce up every time to the same height and strike the platform at always the same level, as indicated by the solid line in Figure 1.12. /37

We note that during the intervals between its impact on the platform the ball is under the influence of only the force of gravity and moves parabolically in accordance with the equation

$$x_2 = -\frac{gt^2}{2} + \dot{x}_{20}t + x_c, \qquad (1.13)$$

where $\dot{x}_{20}$ is the ball's initial velocity at the instant of rebound, x_c is the coordinate characterizing the position of the ball and platform at the instants when they collide.

* A similar system is analyzed in [44, 55, 56].

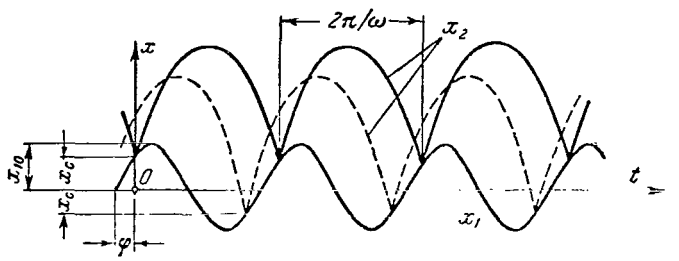

Figure 1.12.

After a complete period of the platform's oscillation $T = \frac{2\pi}{\omega}$ the ball again strikes the platform, with a velocity that is equal to $\dot{x}_{20}$ but is now directed downward.

Differentiating (1.13), we get

$$\dot{x}_2 = -gt + \dot{x}_{20}.$$

Substituting into this $t = 2\pi/\omega$ and $\dot{x} = -\dot{x}_{20}$, we find that

$$\dot{x}_{20} = \frac{\pi g}{\omega}. \tag{1.14}$$

Now we have to fit the motion of the ball and that of the platform together. We shall stipulate that the ball's repeated striking of the platform in no way affects the latter's motion, and that it continues to move harmonically. Since according to (1.13) the time is counted from the instant of the ball's striking the platform, we can now write the law for the platform's motion as: /38

$$x = x_{10}\sin(\omega t + \varphi),$$
$$\dot{x} = x_{10}\omega\cos(\omega t + \varphi),$$

where ϕ is the phase characterizing the instant of impact. Hence the platform's velocity at the instant of impact will be

$$\dot{x}_{10} = x_{10}\omega\cos\varphi \tag{1.15}$$

and (depending on the condition) as a result of the impact will remain unchanged. The ball's velocity $\dot{x}_{20}$ after the impact will reverse sign.

According to (1.10), we can now write

$$R = \frac{\dot{x}_{20} - \dot{x}_{10}}{\dot{x}_{20} + \dot{x}_{10}}.$$

Substituting into this the value of $\dot{x}_{20}$ from (1.14), we get

$$\dot{x}_{10} = \frac{1-R}{1+R}\frac{\pi g}{\omega}.$$

Comparing this equality with (1.15), we find the impact phase, whence the position of the platform at the instant of impact:

$$\cos\varphi = \frac{1-R}{1+R}\frac{\pi g}{x_{10}\omega^2}, \tag{1.16}$$

$$x_c = \pm x_{10}\sqrt{1-\cos^2\varphi}. \tag{1.17}$$

Taking into account that we must have the inequality $\phi \leq 1$, we get

$$x_{10}\omega^2 \geqslant \frac{1-R}{1+R}\pi g.$$

This inequality stipulates the necessary value of the vibration intensity at which the ball's periodic motion can become established. As we see, it is greatly dependent on the magnitude of the restitution coefficient. Thus, for example, if R = 1, i.e., if the ball's impact on the platform is absolutely elastic, then the ball will be moving periodically even if the platform stays motionless.

We return now to equations (1.16) and (1.17). Note that the first of these gives two possible values for the impact phase. Accordingly, (1.17) gives two different values for the coordinate x_c. So we have not obtained a unique solution of the dynamic problem. In other words, with the same set of values for our model's parameters the ball can have two different states of motion. In Figure 1.12 the second state is represented by the broken line. /39

Here we have encountered another property characteristic of nonlinear systems: nonuniqueness. One naturally asks "But which of the two possible states of motion does the system actually realize?" How does the ball really move, as indicated by the solid line or by the broken line? We get the answer by analyzing the stability of both states of motion.

We use for this a method that we already know. We introduce a small perturbation into the ball's periodic motion, then observe the ball's subsequent behavior. If with time its perturbed motion increasingly deviates from its unperturbed, then its state of motion is unstable.

Let us look at the state of motion represented in Figure 1.12 by the broken line, and assume that after a regular impact at a reference point a the ball received a certain perturbation as a result of which its velocity underwent a small variation (Figure 1.13,a). For determinancy we assume that the velocity increased becoming

$$\dot{x}'_{20} > \dot{x}_{20}.$$

Moving at a velocity somewhat greater than its calculated velocity, the ball will strike the platform somewhat later, i.e., at the instant a' instead of a. We note that at the instant a' the platform's velocity is somewhat greater than at the instant a. Hence at the instant of a regular impact

$$\dot{x}'_{10} > \dot{x}_{10}.$$

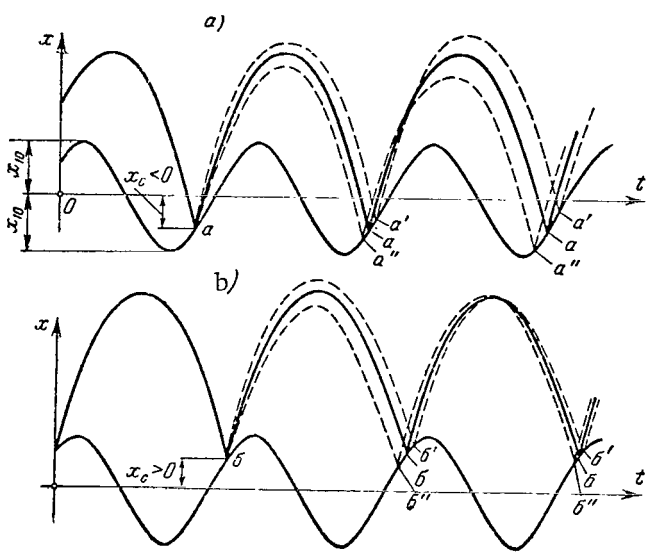

Figure 1.13.

As a result, the ball's rebound velocity is again increased, along with the time of its motion until the next impact. The point of impact will be further shifted; the platform's velocity at the instant of impact will again increase, etc. The ball's perturbed motion will increasingly deviate from the unperturbed. So this state of the ball's motion is unstable. We verify that by assuming that, as a result of the initial perturbation, the ball's velocity became not greater but less than the calculated. /40

Having started to move at the velocity $\dot{x}'_{20} < \dot{x}_{20}$, the ball will strike the platform sooner at ("point a"). Here the platform's velocity will be less than the calculated. As a result, after rebound the ball's velocity will again decrease; the next impact will occur still sooner, etc. So in this case the perturbed motion will increasingly deviate from the unperturbed. Consequently, the states of the ball's motion for which the collision coordinate $x_c < 0$ are unstable.

Now we look at another state of motion, depicted in Figure 1.13,b. Proceeding by the usual method, we introduce at point b an initial perturbation, as a result of which the ball's velocity has again become /41

$$\dot{x}'_{20} > \dot{x}_{20}.$$

True, in this case also the ball will strike the platform somewhat later than in its unperturbed motion (at point b' instead of b). But in the interval from $x_1 = 0$ to $\dot{x}_1 = x_{10}$ the velocity of the platform decreases, whereas in the interval from $x_1 = -x_{10}$ to $x_1 = 0$ it increased. So now the platform's

velocity at the instant of impact will be somewhat less than the calculated. As a result, the ball's velocity will no longer increase but, on the contrary, will decrease, and the introduced perturbation will be to some extent compensated. If the compensation is insufficient and the time of the ball's motion until the next impact increasingly exceeds the calculated, the ball will again strike the platform somewhat later, and the platform's velocity at the instant of impact will again be less. As a result, the ball's rebound velocity will again decrease, etc.

So in this case the effect of the intial perturbation will always be compensated; the perturbed motion will tend toward the unperturbed; hence this state of motion is stable.

We can easily verify that the ball will behave similarly if, as a result of the initial perturbation its velocity decreased rather than increased. Actually, in this case the ball will strike the platform at the instant b". But then the platform's velocity at the instant of impact will exceed the calculated, and after the impact the ball's velocity will increase. By observing the ball's motion through a number of these collisions, we see that its perturbed motion will tend toward the unperturbed. So the state of motion for which $x_c > 0$ is stable.

Later on, in considering the stability of vibroimpact systems, we shall see that there are whole ranges of stable and unstable states of motion, and that unless one knows its stability pattern, one cannot correctly calculate and design a system.

Now we are going to use our model to look into one more problem, precise theoretical solution of which often involves considerable difficulties. We shall assume that the ball's unperturbed motion was characterized by the impact/42 phase $\phi = 0$, $\cos \phi = 1$.

From the equality (1.16) we readily find the corresponding system parameters assuring the possibility of such a state of motion. According to (1.17), we then have $x_c = 0$.

Attempting to analyze the stability of this state of motion, we ascertain the following. If as a result of the initial perturbation the ball's velocity increases, it will move thereafter as though its unperturbed motion were stable. If as a result of the initial perturbation the ball's velocity decreases, it will move thereafter as though its unperturbed motion were unstable. This is the motion's borderline state, coinciding with the line separating the range of its stable from that of its unstable states. Since perturbations acting upon any system can be of a completely random nature, such a borderline state naturally has to be classed with the unstable ones. It is easy to conceive, moreover, that the entire set of the motion's states close to the borderline state may turn out to be unstable, if only for the simple reason that the values of the physical quantities figuring in the calculation are known to us only approximately.

So, in designing mechanisms with elastic couplings as well as vibroimpact systems, we must try to see to it that they have a certain stability margin, i.e., that their calculated motion states are kept sufficiently distant from the boundaries of the unstable-solution regions.

We conclude this section with one more observation. In checking the stability of this or that state of the motion of a two-body vibroimpact system, we proceeded successively from one collision to another and determined the coordinate increments and collision velocities of the two bodies.

We designate those increments as:

$$\begin{aligned} &\Delta x_{c1}, \quad \Delta x_{c2}, \quad \ldots, \quad \Delta x_{cn}, \\ &\Delta \dot{x}_{11}, \quad \Delta \dot{x}_{12}, \quad \ldots, \quad \Delta \dot{x}_{1n}, \\ &\Delta \dot{x}_{21}, \quad \Delta \dot{x}_{22}, \quad \ldots, \quad \Delta \dot{x}_{2n}. \end{aligned}$$

Solving the problem of the stability of the motion of a vibroimpact system then reduces to determining the variation patterns of the quantities Δx_{cn}, $\Delta \dot{x}_{1n}$, $\Delta \dot{x}_{2n}$ for a number of collisions n tending toward infinity. For solution of such problems it is convenient to use a mathematical device based on so-called difference equations, which is why at the end of the next chapter we give the elements of the theory of those equations.

CHAPTER 2

ELEMENTS OF THE THEORY OF DIFFERENTIAL AND DIFFERENCE EQUATIONS

§ 2.1. Linear Differential Equations with Constant Coefficients. /44

A differential equation is a relation linking a certain independent variable x, an unknown function of that variable y(x) and the functions $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, etc.

The order of a differential equation depends on the highest degree of the derivatives that it contains. In a general form a linear differential equation of n-th order can be written as:

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_n(x)\,y = \varphi(x), \tag{2.1}$$

where $a_1(x), \ldots, a_n(x)$, $\phi(x)$ are given functions of the independent variable x. If $\phi(x) = 0$, the equation is called homogeneous, in the opposite case nonhomogeneous. The functions $a_1(x), \ldots, a_n(x)$ are called coefficients of the linear differential equation. If they are constant, then we say that equation (2.1) is a linear differential equation of n-th order with constant coefficients.

Remembering that in this book we shall be dealing with differential equations of no higher than second order, we consider it purposeful to review for the reader only that part of the subject which he needs to know in order to use freely primarily those equations.

A function in the form $y = y(x, C_1, C_2)$, which is the general solution /45 of a second-order nonhomogeneous linear differential equation, can be represented as the sum of a certain particular solution of the equation plus the general solution of the corresponding homogeneous equation. The latter solution is usually sought in the form of a linear combination of two linearly independent functions each satisfying the homogeneous equation.

We shall consider primarily equations with constant coefficients. In a general form a homogeneous second-order equation can be written as:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0. \tag{2.2}$$

The solution of this equation is sought in the form $y = e^{\lambda x}$. Substituting it into the equality (2.2), we get the following characteristic equation:

$$\lambda^2 + a\lambda + b = 0, \tag{2.3}$$

whose roots are

$$\lambda_1 = -\frac{a}{2} + \sqrt{\frac{a^2}{4} - b}; \qquad \lambda_2 = -\frac{a}{2} - \sqrt{\frac{a^2}{4} - b}.$$

Let us look at the three possible cases wherein the roots of the characteristic equation are: 1) real and different, 2) real and identical (multiple, as they are called), and 3) complex.

1. The first case occurs when the relation between the characteristic equation's coefficients is such that $a^2 > 4b$.

Then there exist two linearly independent solutions of equation (2.2), and its general solution is written as:

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}. \tag{2.4}$$

2. The roots of equation (2.3) are multiple on condition that $a^2 = 4b$. In this event we get only one solution, in the form $Ce^{\lambda x}$, where $\lambda = -a/2$. But we know that the general solution has to contain two arbitrary constants. So there must exist a second particular solution differing from the one just indicated. It is sought in the following form: /46

$$y = ue^{-\frac{a}{2}x}, \tag{2.5}$$

where u = u(x) is an initially unknown function of x.

By substituting this expression into (2.2) the reader will be able to verify that the function u(x) has the property

$$\frac{d^2u}{dx^2} = 0,$$

whence it follows that it is a linear function of x. Specifically we can select

$$u = x. \tag{2.6}$$

Substituting this value into (2.5), we get the general solution in the form

$$y = (C_1 + C_2 x)\, e^{-\frac{a}{2}x} \tag{2.7}$$

3. In the case wherein $a^2 < 4b$ the roots of the characteristic equation are complex. This is the case that we shall most frequently have to deal with. Since complex roots are always conjugate, the general solution can be written in the form

$$y = c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha-\beta i)x}, \tag{2.8}$$

where

$$\alpha = -\frac{a}{2}, \qquad \beta = \sqrt{b - \frac{a^2}{4}}.$$

By virtue of the Euler formulae

$$e^{ix} = \cos x + i \sin x,$$
$$e^{-ix} = \cos x - i \sin x$$

expression (2.8) can be rewritten as:

$$y = e^{-\frac{a}{2}x}\left(C_1 \cos \sqrt{b^2 - \frac{a^2}{4}}\, x + C_2 \sin \sqrt{b - \frac{a^2}{4}}\, x\right) \tag{2.9}$$

or /47

$$y = De^{-\frac{a}{2}x} \cos\left(\sqrt{b - \frac{a^2}{4}}\, x - E\right), \tag{2.10}$$

where

$$D = \sqrt{C_1^2 + C_2^2}, \quad \tan E = \frac{C_2}{C_1} \begin{cases} \sin E = \frac{C_2}{D}, \\ \cos E = \frac{C_1}{D}. \end{cases}$$

Now we can turn our attention to nonhomogeneous differential equations of the form

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x). \tag{2.11}$$

As already pointed out, to find the general solution, it remains for us to find any particular solution of equation (2.11).

The function f(x) can have any form, but here we are confining ourselves to the particular case wherein this function is harmonic.

We shall find a particular solution of the equation in the form

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = F_0 \cos x. \tag{2.12}$$

The standard procedure for seeking this solution consists in determining the coefficients A and B of the function

$$y = A\cos x + B\sin x \tag{2.13}$$

in such a way that the function satisfies equation (2.12). Doubly differentiating expression (2.13) and substituting the found relationships in (2.12), we get

$$[(b-1)A + aB]\cos x + [(b-1)B - aA]\sin x = F_0 \cos x.$$

This equation holds for any values of x when the following equalities are fulfilled:

$$\left.\begin{aligned} (b-1)\,A + aB &= F_0, \\ (b-1)\,B - aA &= 0. \end{aligned}\right\} \tag{2.14}$$

Solving them jointly, we find

$$A = \frac{(b-1)F_0}{(b-1)^2+a^2}; \qquad B = \frac{aF_0}{(b-1)^2+a^2}. \tag{2.15}$$

Now the particular solution (2.13) can be written as /48

$$y = \frac{F_0}{(b-1)^2+a^2}[(b-1)\cos x + a\sin x],$$

or as

$$y = \frac{F_0}{\sqrt{(b-1)^2+a^2}}\cos(x-\alpha), \quad \tan\alpha = \frac{a}{b-1} \left\{\begin{aligned} \sin\alpha &= a, \\ \cos\alpha &= b-1. \end{aligned}\right.$$

The complete solution of equation (2.12) consists of the just found particular solution and one of the solutions (2.4), (2.7) or (2.9), depending on the

values of the roots of the characteristic equation (2.3).

In problems of dynamics, conditions analogous to the condition b = 1 are usually called resonance conditions.

§ 2.2. Linear Differential Equations with Periodic Coefficients.

It was shown in the introduction that a number of problems of the dynamics of mechanisms with elastic couplings necessitate analysis of differential equations with periodic coefficients. The theory of these equations is much more complicated than that of equations with constant coefficient. Naturally, in setting forth the elements of this theory we shall, as before, confine ourselves to second-order equations. We start by seeking the general solution of a homogeneous equation in the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \tag{2.16}$$

whose coefficients $p(x) = p(x + \pi)$ and $q(x) = q(x + \pi)$ are continuous periodic functions of an independent variable having the same period. Taking this period as equal to π in no way violates the generality of the result.

The method of seeking a solution for equation (2.16) reduces to the following. We regard as known two linearly independent solutions of this equation, $y_1 = y_1(x)$ and $y_2 = y_2(x)$, which form the so-called fundamental system. The coefficients $p(x)$ and $q(x)$ being periodic, equation (2.16) will not change if $(x + \pi)$ is substituted for x. Hence the function $y_1^* = y_1(x + \pi)$ /49 and $y_2^* = y_2(x + \pi)$, obtained from y_1 and y_2 by substituting $(x + \pi)$ for x, are also particular solutions of equation (2.16).

The functions y_1^* and y_2^* can be expressed linearly in terms of y_1 and y_2:

$$y_1^* = \alpha_1 y_1 + \alpha_2 y_2; \qquad y_2^* = \beta_1 y_1 + \beta_2 y_2, \tag{2.17}$$

where α_1, α_2, β_1, β_2 are constants.

So, when $(x + \pi)$ is substituted for x, the fundamental system y_1, y_2 is transformed into the fundamental system (2.17).

Instead of the fundamental system y_1 and y_2 we introduce another fundamental system Y_1 and Y_2, selecting it so that, when $(x + \pi)$ is substituted for x, the linear transformation will have the simplest possible form. To do this, we determine the two constants γ_1 and γ_2 so that, when $(x + \pi)$ is substituted for x, the function $Y = Y(x) = \gamma_1 y_1 + \gamma_2 y_2$ will assume the form

$$Y^* = Y(x + \pi) = \phi Y, \tag{2.18}$$

where ϕ is the constant to be determined. Hence we should have the equality

$$\gamma_1 y_1^* + \gamma_2 y_2^* = \varphi(\gamma_1 y_1 + \gamma_2 y_2).$$

Substituting the values of y_1^* and y_2^* from equations (2.17), we get

$$\gamma_1(\alpha_1 y_1 + \alpha_2 y_2) + \gamma_2(\beta_1 y_1 + \beta_2 y_2) - \varphi(\gamma_1 y_1 + \gamma_2 y_2) = 0$$

or

$$[(\alpha_1 - \varphi)\gamma_1 + \beta_1\gamma_2]\, y_1 + [\alpha_2\gamma_1 + (\beta_2 - \varphi)\gamma_2] y_2 = 0.$$

Since the solutions y_1 and y_2 are linearly independent, the transformation (2.18) is fulfilled if

$$\left.\begin{aligned}(\alpha_1 - \varphi)\gamma_1 + \beta_1\gamma_2 = 0,\\ \alpha_2\gamma_1 + (\beta_2 - \varphi)\gamma_2 = 0.\end{aligned}\right\} \tag{2.19}$$

For values of γ_1 and γ_2 different from zero the determinant of this homogeneous system should be equal to zero: /50

$$\begin{vmatrix}(\alpha_1 - \varphi) & \beta_1\\ \alpha_2 & (\beta_2 - \varphi)\end{vmatrix} = \varphi^2 - (\alpha_1 + \beta_2)\varphi + (\alpha_1\beta_2 - \beta_1\alpha_2). \tag{2.20}$$

The roots ϕ_1 and ϕ_2 of this characteristic equation enable us to determine those values of the constants $\gamma_{11}, \gamma_{21}, \gamma_{12}, \gamma_{22}$ for which the particular solutions

$$Y_j = \gamma_{j1} y_1 + \gamma_{j2} y_2 \qquad (j = 1, 2)$$

resulting from substitution of (x + π) for x assume the form

$$Y_j^* = \varphi_j Y_j. \tag{2.21}$$

Writing the roots ϕ_j of the characteristic equation in the form

$$\phi_j = e^{\mu_i \pi}$$

and multiplying the left- and right-hand sides of equations (2.21) by $e^{-(x+\pi)\mu_j}$, we get

$$e^{-(x+\pi)\mu_j} Y_j(x+\pi) = e^{-\mu_j x} Y_j(x) = \Phi_j(x).$$

The functions $\Phi_j(x)$, each of which is the product of two functions, do not change when $(x + \pi)$ is substituted for x, hence are periodic functions having the period π.

From these equalities it follows that the particular solutions of a homogeneous equation with periodic coefficients will have the form

$$Y_j(x) = e^{\mu_j x} \Phi_j(x)$$

whence the general solution, in the case of different roots of the characteristic equation, will be:

$$y = C_1 e^{\mu_1 x} \Phi_1(x) + C_2 e^{\mu_2 x} \Phi_2(x). \tag{2.22}$$

We shall establish the character of the general solution in the case of multiple roots of the characteristic equation. The roots become multiple ($\phi_j = \phi_0$) when $\alpha_2 = 0$ or $\beta_1 = 0$ or $\alpha_2 = \beta_1 = 0$ [see equation (2.20)]. We let, for example, $\alpha_2 = 0$. Then the equalities (2.19) are satisfied when $\gamma_1 = \beta_1, \gamma_2 = \phi_0 - \alpha_1$.

Consequently, there is only one solution in the form (2.21): /51

$$Y_1 = \gamma_1 y_1 + \gamma_2 y_2 = \beta_1 y_1 + (\varphi_0 - \alpha_1) y_2, \tag{2.23}$$

which, when $(x + \pi)$ is subsituted for x, undergoes the required transformation $Y_1^* = \phi_0 Y_1$. We select the second, linearly independent solution in the form $Y_2 = y_2$. Then, when $(x + \pi)$ is substituted for x, we get

$$Y_2^* = y_2^* = \beta_1 y_1 + \beta_2 y_2.$$

Substituting in place of $\beta_1 y_1$ its values from expression (2.23), we get

$$Y_2^* = Y_1 + (\alpha_1 + \beta_2 - \varphi_0) Y_2 = \varphi_0 Y_2 + Y_1.$$

Hence in the case of multiple roots the fundamental system will have the form:

$$Y_1^* = \varphi_0 Y_1; \qquad Y_2^* = \varphi_0 Y_2 + Y_1.$$

As before, we write the multiple root ϕ_0 as $\phi_0 = e^{\mu 0 \pi}$.

So, if the roots of a characteristic equation are multiple, the first particular solution will be

$$Y_1 = e^{\mu_0 x} \Phi_1(x),$$

where $\Phi_1(x)$ is a periodic function having the period π. We can write the solution Y_2 in a form giving a clear picture of the nature of this solution by using the function

$$\Phi_2(x) = \frac{Y_2}{e^{\mu_0 x}} - \frac{x}{\varphi_0 \pi} \Phi_1(x).$$

Now Y_2 assumes the form

$$Y_2 = e^{\mu_0 x} \left[\Phi_2(x) + \frac{x}{\varphi_0 \pi} \Phi_1(x) \right]$$

When $(x + \pi)$ is substituted for x, the function $\Phi_2(x)$ becomes

$$\Phi_2(x + \pi) = \frac{\varphi_0 Y_2 + Y_1}{\varphi_0 e^{\mu_0 x}} - \frac{x + \pi}{\varphi_0 \pi} \Phi_1(x) = \Phi_2(x)$$

whence, like $\Phi_1(x)$, it too is a periodic function having the period π. /52

So, in the case of multiple roots of the characteristic equation, the general solution of the homogeneous equation has the character:

$$Y = e^{\mu_0 x} [C_1 x \Phi_1(x) + C_2 \Phi_2(x)].$$

Comparing this solution to solution (2.7) for the case of multiple roots, with constant coefficients of the differential equation, we discover an important difference. Now the expression in the square brackets is not a simple linear function of x.

§ 2.3. The Mathieu Equation.

We write this equation thus:

$$\frac{d^2y}{dx^2} + (a - 2q\cos 2x)\, y = 0. \tag{2.24}$$

Using the above theory of equations with periodic coefficients, we shall find the general functional character of the solution of the Mathieu equation. We regard as known two particular, linearly independent solutions y_1 and y_2 of equation (2.24) which form the fundamental system satisfying the initial conditions, when x = 0:

$$\begin{aligned} y_1(0) &= 1; \quad y_1'(0) = 0; \\ y_2(0) &= 0, \quad y_2'(0) = 1. \end{aligned}$$

Differentiating the expressions (2.17) and designating $\frac{dy}{dx} = y'$, we get

$$\left.\begin{aligned} y_1'(x+\pi) &= y_1^{*\prime} = \alpha_1 y_1' + \alpha_2 y_2', \\ y_2'(x+\pi) &= y_2^{*\prime} = \beta_1 y_1' + \beta_2 y_2'. \end{aligned}\right\} \tag{2.25}$$

Utilizing the initial conditions, from equations (2.17) and (2.25) we determine the values of the constants α_1, α_2, β_1, β_2:

$$\alpha_1 = y_1(\pi); \quad \alpha_2 = y_1'(\pi); \quad \beta_1 = y_2(\pi); \quad \beta_2 = y_2'(\pi).$$

Now the characteristic equation (2.20) will be:

$$\varphi^2 - [y_1(\pi) + y_2'(\pi)]\,\varphi + [y_1(\pi)\, y_2'(\pi) - y_2(\pi)\, y_1'(\pi)] = 0.$$

The free term in this equation is the determinant, which with p(x) ≡ 0 remains constant for any value of x (see [66]). Under the selected initial conditions this determinant is /53

$$\delta = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = 1$$

whence the characteristic equation assumes the form

$$\varphi^2 - [y_1(\pi) + y_2'(\pi)]\,\varphi + 1 = 0.$$

Expressing y_1 in terms of y_1^* and y_2^* by virtue of (2.17), we get

$$y_1(x) = y_2'(\pi) y_1(x+\pi) - y_1'(\pi) y_2(x+\pi).$$

Substituting in this the value x = -π, we get

$$y_1(-\pi) = y_2'(\pi).$$

The periodic coefficient of equation (2.24),

$$q(x) = a - 2q \cos 2x$$

is an even function, whence the characteristic equation simplifies, assuming the form

$$\varphi^2 - 2y_1(\pi)\varphi + 1 = 0.$$

According to Vieth's theorem*

$$\varphi_1\varphi_2 = 1; \quad \varphi_1 + \varphi_2 = 2y_1(\pi).$$

If we now set

$$\varphi_1 = e^{\mu\pi}, \quad \varphi_2 = e^{-\mu\pi},$$

we get

$$\frac{e^{\mu\pi} + e^{-\mu\pi}}{2} = \operatorname{ch}\mu\pi = y_1(\pi).$$

The evenness of the periodic coefficient enables us to write the general solution of the Mathieu equation in the form

$$y = Ae^{\mu x}\Phi(x) + Be^{-\mu x}\Phi(-x), \tag{2.26}$$

where Φ(x) is a periodic function (with a period π), μ a characteristic exponent whose value depends on the constant parameters a and q of the Mathieu /54 equation and not on the initial conditions.

The solution (2.26) is called stable if it remains finite when $x \to \infty$,

* The name here is uncertain. The phonetic Russian spelling makes it sound like "Viet" or "Viyet." Tr.

unstable if it infinitely increases when $x \to \infty$.

Since in this solution the function $\Phi(x)$ is periodic while A and B are constants, its stability is obviously entirely determined by the value of the characteristic exponent μ, i.e., by the values of the roots ϕ_1 and ϕ_2.

The roots of the characteristic equation can be:

1) real, different and positive if

$$y_1(\pi) = \operatorname{ch} \mu\pi > 1.$$

in which case the characteristic exponent μ will be a real quantity;

2) real, different and negative if

$$y_1(\pi) = \operatorname{ch} \mu\pi < -1.$$

in which case the characteristic exponent μ will be complex, having the form $\mu = \alpha + in$, $n = 0, 1, 2, \ldots$; [in both these cases the solution (2.26) is unstable.]

3) complex and conjugate if

$$-1 < y_1(\pi) = \operatorname{ch} \mu\pi < 1.$$

in which case the characteristic exponent $\mu = i\beta$ will be imaginary, and solution (2.26) will be stable and can be written as

$$y = Ae^{-i\beta x}\Phi(x) + Be^{-i\beta x}\Phi(-x).$$

If β is a simple rational fraction of the form n/m, then both particular solutions are periodic functions having the period $2\pi m$. If β is an irrational quantity, both solutions, remaining finite, will be aperiodic.

In cases in which

$$y_1(\pi) = \pm 1,$$

the roots of the characteristic equation are multiple: /55

$$\varphi_1 = \varphi_2 = \varphi_0 = \pm 1.$$

The values of the characteristic exponent μ for $\phi_0 = +1$ and $\phi_0 = -1$

are determined from the condition

$$e^{\mu\pi} = +1,$$

whence $\mu = i\beta = i2n$, where $n = 0, 1, 2, \ldots$, and from the condition $e^{\mu\pi} = -1$, whence $\mu = i\beta = 1(2n + 1)$, where $n = 0, 1, 2, \ldots$

We mentioned earlier that in the case of multiple roots only one particular solution can be written in the form

$$y_1 = e^{i\beta x}\Phi(x).$$

If here $\beta = 2n$, when $(x + \pi)$ is substituted for x, the value and sign of this particular solution will remain unchanged:

$$e^{i2n(x+\pi)}\Phi(x+\pi) = e^{i2nx}\Phi(x) = f_1(x),$$

where $f_1(x)$ is a periodic solution whose period is equal to π, i.e., to the period of the function $\Phi(x)$.

If $\beta = 2n + 1$, when $(x + \pi)$ is substituted for x, the particular solution's value will remain the same, but its sign will change:

$$e^{i(2n+1)(x+\pi)}\Phi(x+\pi) = e^{i(2n+1)x}\Phi(x),$$

and only after substitution of $(x + 2\pi)$ for x do we have fulfillment of the equality

$$e^{i(2n+1)(x+2\pi)}\Phi(x+2\pi) = e^{i(2n+1)x}\Phi(x) = f_2(x),$$

where $f_2(x)$ is a periodic solution whose period is equal to 2π.

Periodic solutions having the period π are sometimes called simply periodic, those with periods 2π semiperiodic.

So in cases of multiple roots one of the particular solutions of the Mathieu equation is periodic, having the period π or 2π.

In accordance with what we have said above, with $\beta = 2n$ the second particular solution will be /56

$$Y_2 = f_3(x) + Cxf_1(x),$$

and with $\beta = 2n + 1$ it will be

$$y_2 = f_4(x) + Cxf_2(x),$$

C being a constant.

The functions $f_3(x)$ and $f_4(x)$ have the periods π and 2π, respectively. The presence of the factor x with the periodic functions $f_1(x)$ and $f_2(x)$ indicates that in the case of multiple roots the second particular solutions are always unstable.

Consequently, the general solution $y = Ay_1 + By_2$ in this case is unstable also.

The periodic and semiperiodic solutions of the Mathieu equation, which have an important meaning in different applications, have been given the name "Mathieu functions of the first kind."

The Mathieu equation has periodic solutions only in the case in which the parameters a and q [see equation (2.24)] are related in a certain way. This relation between a and q is written in the form of a series

$$a = m^2 + a_1q + a_2q^2 + a_3q^3 + \ldots, \tag{2.27}$$

where a_1, a_2, etc., are constant coefficients to be determined.

The periodic solutions are also written as a series

$$y = \cos mx + qc_1(x) + q^2c_2(x) + q^3c_3(x) + \ldots, \tag{2.28}$$

or

$$y = \sin mx + qs_1(x) + q^2s_2(x) + q^3s_3(x) + \ldots, \tag{2.29}$$

where $c_i(x)$ and $s_i(x)$, which are independent of q, are periodic functions of x to be determined.

Substituting the series (2.27) and (2.28) or (2.27) and (2.29) in the Mathieu equation, and bringing together the terms containing the same powers of q, we get a system of linear differential equations with constant coefficients, and with that system we determine a_i and $c_i(x)$ or a_i and $s_i(x)$.

Substituting the found values of a_i and of the function $c_i(x)$, $s_i(x)$ in /57 the respective series, we obtain the sought-for periodic solutions, i.e., the Mathieu functions.

In the case of (2.28), if m = 2n is any even number, the solutions

obtained are even and periodic. If m = 2n + 1 is an odd number, the solutions obtained are even and semiperiodic. In the case of (2.29), if m is even, the solutions obtained are odd and periodic; if m is odd, they are odd and semiperiodic.

By this method we can determine an infinite series of even and odd Mathieu functions, which we designate as:

$$ce_0(x, q),\quad ce_1(x, q),\quad ce_2(x, q),\ \ldots,\ ce_m(x, q),$$
$$se_1(x, q),\quad se_2(x, q),\ \ldots,\ se_m(x, q).$$

When m = 0, there is only one even Mathieu function whose period is equal to π [the function $ce_0(x, q)$].

Usually all Mathieu functions of the first kind reduce to four types, which can be written as the following trigonometric functions of the independent variable x:

$$ce_{2n}(x, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos 2rx \; -$$ even Mathieu functions (of period π);

$$ce_{2n+1}(x, q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cos(2r+1)x \; -$$ even Mathieu functions (of period 2π);

$$se_{2n+1}(x, q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sin(2r+1)x \; -$$ odd Mathieu functions (of period 2π;

$$se_{2n+2}(x, q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sin(2r+2)x \; -$$ odd Mathieu functions (of period π).

The indices 2n, 2n + 1 and 2n + 2 indicate the number of zeros of the function in question in the interval $0 < x < \frac{\pi}{2}$, which designates the function's order. The subscripts 2r, 2r + 1 and 2r + 2 pertain directly to the coefficients A and B of the respective functions, the values of which coefficients depend on the parameter q.

Each of the Mathieu functions corresponds to a specific relation between the parameters a and q of the equation. This relation is expressed by the series (2.27), and the numbers a, calculated for different values of q, are called characteristic numbers of the Mathieu functions. /58

We shall designate as a_m characteristic numbers of the functions $ce_m(x, q)$, as b_m those of the functions $se_m(x, q)$.

Table 2.1 gives rounded-off values of the numbers $a_0, \ldots, a_5$ and $b_1, \ldots, b_5$ within the limits q = 0-14.

TABLE 2.1. CHARACTERISTIC NUMBERS OF THE FUNCTIONS $ce_m(x, q)$, m = 0-5 and $se_m(x, q)$, m = 1-5 /59

q	a_0	b_1	a_1	b_2	a_2	b_3	a_3	b_4	a_4	b_5	a_5
0	0,000	1,000	1,000	4,000	4,000	9,000	9,000	16,000	16,000	25,000	25,000
1	—0,455	—0,110	1,859	3,917	4,371	9,048	9,078	16,003	16,034	25,021	25,021
2	—1,514	—1,390	2,379	3,672	5,173	9,141	9,370	16,128	16,141	25,083	25,084
3	—2,834	—2,185	2,519	3,277	6,045	9,223	9,916	16,273	16,339	25,187	25,190
4	—4,281	—4,259	2,318	2,747	6,829	9,261	10,671	16,452	16,650	25,331	25,344
5			1,858	2,099	7,449	9,236	11,549	16,648	17,097	25,511	25,550
6			1,214	1,351	7,870	9,138	12,466	16,845	17,689	25,723	25,217
7			0,438	0,518	8,087	8,962	13,358	17,027	18,417	25,962	26,156
8			—0,436	—0,389	8,115	8,710	14,182	17,183	19,253	26,221	26,578
9			—1,387	—1,359	7,983	8,383	14,904	17,303	20,161	26,492	27,092
10			—2,399	—2,382	7,717	7,986	15,503	17,381	21,105	26,766	27,704
11			—4,570	—4,563	6,879	7,001	16,301	17,395	22,972	27,300	29,208
12			—	—	5,736	5,793	16,599	17,207	24,651	27,770	31,000

Tr. Note: Commas indicate decimal points.

Figure 2.1 gives the curves of the characteristic numbers $a_m = a_m(q)$, $b_m = b_m(q)$. The curves a_0, b_2, a_2, b_4, a_4 are symmetrical with respect to the axis a, while the curves b_1, a_1, b_3, a_3, b_5, a_5 are not. Nevertheless, /60 the graph as a whole is symmetrical, because b_1 and a_1, b_3 and a_3, b_5 and a_5, respectively, do possess mutual symmetry with respect to the a-axis.

As q increases (with a > 0), the curves a_m and b_{m+1} draw rapidly closer together (without actually touching). As the negative values of q increase, the curves a_m and a_{m+1}, b_m and b_{m+1} draw closer together.

In both cases one of the converging curves is for the function having the period π, the other for the function having the period 2π.

The lines of the characteristic numbers of the Mathieu functions divide the plane a, q into a series of regions or zones. We show below that, depending on which of these regions the Mathieu equation's characteristic point (a, q) falls in, the equation's solution will be stable or unstable. For this reason Figure 2.1 is often referred to as a stability "chart."

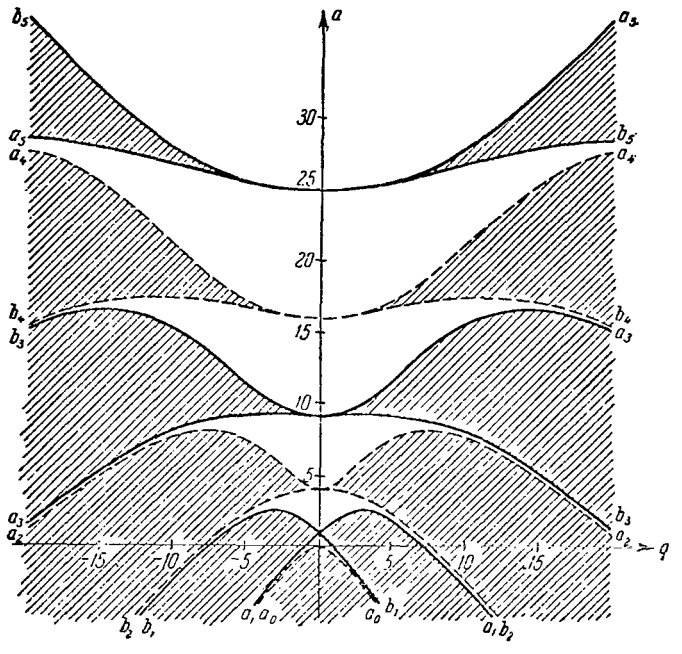

Figure 2.1.

§ 2.4. Relation Between μ, a and q in the Mathieu Equation.

If the periodic function Φ(x) in (2.26) is expanded into a Fourier series by our writing it in complex form, one of the particular solutions of the fundamental system becomes

$$y_1 = e^{\mu x} \sum_{r=-\infty}^{\infty} c_{2r} e^{2rix}. \qquad (2.30)$$

With given values of the parameters a and q, in order to find a solution of the equation, we have to determine the value of the characteristic exponent μ and the values of the coefficients c_{2r}. Most difficult is finding μ = μ(a, q).

We confine ourselves to indicating only one of the methods (Hill method) for establishing the relation between the Mathieu equation's parameters a and q and the characteristic exponent μ (see [107]).

We substitute in the Mathieu equation the particular solution (2.30), for which purpose we first find d^2y/dx^2 and the other two terms of the equation: /61

$$\frac{d^2y}{dx^2} = \mu e^{\mu x} \sum_{r=-\infty}^{\infty} c_{2r}(2ri+\mu) e^{2rix} +$$

$$+ e^{\mu x} \sum_{r=-\infty}^{\infty} 2ric_{2r}(2ri+\mu) e^{2rix} = -e^{\mu x} \sum_{r=-\infty}^{\infty} c_{2r}(2r-i\mu)^2 e^{2rix},$$

$$ay = e^{\mu x} \sum_{r=-\infty}^{\infty} ac_{2r} e^{2rix},$$

$$(-2q\cos 2x)\, y = -e^{\mu x} \sum_{r=-\infty}^{\infty} qc_{2r}(e^{2ix} + e^{-2ix}) e^{2rix}.$$

Substituting these expressions in the Mathieu equation, we get

$$e^{\mu x} \sum_{r=-\infty}^{\infty} c_{2r} e^{2rix} \{a - (2r-i\mu)^2 - q(e^{2ix} + e^{-2ix})\} = 0.$$

Since we always have $e^{\mu x} \neq 0$, we get

$$\sum_{r=-\infty}^{\infty} c_{2r} e^{2rix} \{a - (2r - i\mu)^2 - q(e^{2ix} + e^{-2ix})\} = 0.$$

We write out three successive terms of this infinite summation

$$\begin{aligned} & c_{2(r-1)} e^{2(r-1)ix} \{a - (2r - 2 - i\mu)^2 - q(e^{2ix} + e^{-2ix})\} + \\ & + c_{2r} e^{2rix} \{a - (2r - i\mu)^2 - q(e^{2ix} + e^{-2ix})\} + \\ & + c_{2(r+1)} e^{2(r+1)ix} \{a - (2r + 2 - i\mu)^2 - q(e^{2ix} + e^{-2ix})\}. \end{aligned}$$

Grouping the coefficients with e^{2rix}, we obtain

$$e^{2rix} \{c_{2r} [a - (2r - i\mu)^2] - q [c_{2(r-1)} + c_{2(r+1)}]\} = 0.$$

Since $e^{2rix} \neq 0$, we have

$$c_{2r} [a - (2r - i\mu)]^2 - q [c_{2(r-1)} + c_{2(r+1)}] = 0.$$

Since the integer r can assume any values between the limits $-\infty$ and $+\infty$, we get as a result a system of an infinite number of linear homogeneous equations for the coefficients c_{2r}, a system written in the form of a so-called recursion formula.

Of special interest is the case in which the characteristic point is in the region of the stable solutions. Then $\mu = i\beta$, and the recursion formula assumes the form /62

$$[a - (2r + \beta)^2] c_{2r} - q [c_{2(r-1)} + c_{2(r+1)}] = 0. \qquad (2.31)$$

Simple analysis tells us how rapidly the coefficients c_{2r} of solution (2.30) decrease as r increases. Substituting in the recursion formula (2.31) the quantity (r + 1) in place of r, we get

$$[a - (2r + 2 + \beta)^2] c_{2r+2} - q [c_{2r} + c_{2r+4}] = 0.$$

We divide this equality by c_{2r}, designating $p_{2r} = \frac{c_{2r+2}}{c_{2r}}$. Accordingly, we designate $p_{2r+2} = \frac{c_{2r+4}}{c_{2r+2}}$. Equation (2.31) can now be written as

$$[a-(2r+2+\beta)^2]\,p_{2r}-q\,[1+p_{2r}p_{2r+2}]=0,$$

whence

$$p_{2r+2}+\frac{1}{p_{2r}}=\frac{[a-(2r+2+\beta)^2]}{q}.$$

When $r \to \infty$, the right-hand side of this equality tends toward $-\infty$:

$$p_{2r+2}+\frac{1}{p_{2r}}\longrightarrow-\infty.$$

Hence either $|p_{2r}| \to 0$ or $|p_{2r}| \to \infty$. Since we are considering the region of stable solutions, when $r \to \infty$, the values of the coefficients $c_{2r} \to 0$. Then, for sufficiently large values of r, we get the inequalities:

$$p_{2r}p_{2r+2} \ll 1, \quad a \ll (2r+2+\beta)^2.$$

Thus, when $r \to \infty$, we have

$$|p_{2r}|=\left|\frac{c_{2r+2}}{c_{2r}}\right| \approx \frac{q}{(2r+2+\beta)^2}\longrightarrow 0.$$

It is readily observed that the values of the parameters a and q exert substantial influence on the rate at which the coefficients c_{2r} decrease, and on their absolute values. If each of the parameters a and q is small compared to unity, it then usually suffices to allow for but a very small number of the coefficients, since their values rapidly decrease as r increases. /63

Coming back to system (2.31), we see that it has a solution only when its determinant equals zero. Dividing each equation by the corresponding coefficient $[a - (2r - i\mu)^2]$, we can set up the so-called Hill equation, which is an infinite determinant with respect to $(i\mu)$:

$$\Delta(i\mu)=\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{q}{a-(-2-i\mu)^2} & 1 & \frac{q}{a-(-2-i\mu)^2} & 0 & 0 \\ 0 & \frac{q}{a-\mu^2} & 1 & \frac{q}{a-\mu^2} & 0 \\ 0 & 0 & \frac{q}{a-(2-i\mu)^2} & 1 & \frac{q}{a-(2-i\mu)^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}=0.$$

This infinite determinant has been calculated by Whittaker and Watson [75].* Its value is

$$\Delta(i\mu) \equiv \Delta(0) - \sin^2\left(\frac{\pi}{2} i\mu\right) \csc^2\left(\frac{\pi}{2}\sqrt{a}\right).$$

So, for determining the value of the characteristic exponent, we have at our disposal the equation $\Delta(i\mu) = 0$, whence

$$\sin^2\left(\frac{\pi}{2} i\mu\right) = \Delta(0) \sin^2\left(\frac{\pi}{2}\sqrt{a}\right), \tag{2.32}$$

where $\Delta(0)$ is the determinant obtained from $\Delta(i\mu)$ when the value $\mu = 0$ is substituted, i.e.,

$$\Delta(0) = \begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{q}{a-4} & 1 & \frac{q}{a-4} & 0 & 0 \\ 0 & \frac{q}{a} & 1 & \frac{q}{a} & 0 \\ 0 & 0 & \frac{q}{a-4} & 1 & \frac{q}{a-4} \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

In calculating the determinant $\Delta(0)$ we confine ourselves to a finite number of terms. For values of q small compared to unity the value of $\Delta(0)$ is determined with the following relation (see [73]) allowing for the terms containing q^2: /64

$$\Delta(0) \approx 1 - \frac{\pi q^2 \operatorname{ctg}\left(\frac{\pi}{2}\sqrt{a}\right)}{4\sqrt{a}(a-1)}.$$

Substituting this value in equation (2.32), we get the sought-for relation between the characteristic exponent μ and the parameters a and q of the Mathieu equation:

$$\cos(i\mu\pi) = \operatorname{ch}(\mu\pi) \approx \cos(\pi\sqrt{a}) + \frac{\pi q^2 \sin(\pi\sqrt{a})}{4\sqrt{a}(a-1)}. \tag{2.33}$$

This relation holds on condition that $a \neq m^2$, m being an integer. Comparing expression (2.33) with the stability condition in the solution of the Mathieu equation (see page34), we are able to establish that in the case in question

* Spelling of authors' names uncertain. The reference appears to be to a translation into Russian from English.

(i.e., when the q are small) the solution will be stable only if the relation between the parameters a and q satisfies the condition

$$-1 < \cos(\pi\sqrt{a}) + \frac{\pi q^2 \sin(\pi\sqrt{a})}{4\sqrt{a}(a-1)} < 1. \tag{2.34}$$

We explained about that, if the point characterized by the given parameters a and q lies on any of the characteristic-number curves a_m or b_m (see Figure 2.1), one of the particular solutions will be periodic or semiperiodic (Mathieu functions), while the other will be unstable, so that the general solution will be unstable also.

Condition (2.34) enables us to determine μ and thus to ascertain the stability or instability in a case in which the characteristic point (a, q) is inside any of the regions bounded by the lines a_m and b_m (q here having to be sufficiently small).

If, with q sufficiently small, we assign to the parameter a different values between the limits $-\infty$ and $+\infty$, we find that:

1) For characteristic points situated below the line a_0 but also between /65 the lines $b_1 - a_1$, $b_2 - a_2$, $b_3 - a_3$, etc., the characteristic exponent is a real or complex number. The solution of the equation is then unstable.

2) For characteristic points situated between the lines $a_0 - b_1$, $a_1 = b_2$, $a_2 - b_2$, etc., we have $-1 < \operatorname{ch} \mu\pi < 1$. The characteristic exponent is then imaginary, and the equation's solution is stable.

Hence the zones (or regions) bounded by two adjacent curves for two periodic or two semiperiodic solutions of the Mathieu equation are zones of unstable solutions or instability zones. Zones bounded by two adjacent curves for one periodic and one semiperiodic solution, respectively, are zones of stable solutions or stability zones. In Figure 2.1 the instability zones are hatched.

§ 2.5. Nonhomogeneous Mathieu Equation.

A nonhomogeneous Mathieu equation is the differential equation

$$\frac{d^2y}{dx^2} + (a - 2q\cos 2x)\,y = f(x), \tag{2.35}$$

whose corresponding homogeneous equation is (2.24). Here f(x) is a periodic function whose period we designate as T, its corresponding angular frequency as p.

As usual, we shall examine a very simple case, considering f(x) a simple harmonic function:

$$f(x) = be^{-ipx}.$$

Utilizing the fact that for linear equations with periodic coefficients the superposition principle operates, we can find the general solution of equation (2.35) by applying the variation-of-constants method [70]: /66

$$y = y_2(x)\int^x \frac{y_1(x)f(x)}{\delta}dx - y_1(x)\int^x \frac{y_2(x)f(x)}{\delta}dx, \qquad (2.36)$$

where $y_1(x)$ and $y_2(x)$ are particular solutions of the Mathieu equation having the form

$$y_1(x) = e^{\mu x}\Phi(x) = e^{\mu x}\sum_{r=-\infty}^{\infty} c_{2r}e^{2rix},$$

$$y_2(x) = e^{-\mu x}\Phi(-x) = e^{-\mu x}\sum_{r=-\infty}^{\infty} c_{2r}e^{-2rix}.$$

The quantity δ in solution (2.36) is the determinant, equal to

$$\delta = y_1(x)\,y_2'(x) - y_2(x)\,y_1'(x) = \text{const} \neq 0.$$

Substituting the expressions for $y_1(x)$, $y_2(x)$ and f(x) in (2.36), after integrating we find:

$$y = \frac{b}{\delta}\left[e^{-\mu x}\sum_{r=-\infty}^{\infty} c_{2r}e^{-2rix}\int^x\left(e^{\mu x}e^{ipx}\sum_{r=-\infty}^{\infty} c_{2r}e^{2rix}\right)dx - \right.$$
$$\left. - e^{\mu x}\sum_{r=-\infty}^{\infty} c_{2r}e^{2rix}\int^x\left(e^{-\mu x}e^{ipx}\sum_{r=-\infty}^{\infty} c_{2r}e^{-2rix}\right)dx\right].$$

As a result of the integration, the solution assumes the form:

$$y = Ae^{\mu x}\sum_{r=-\infty}^{\infty} c_{2r}e^{2rix} + Be^{-\mu x}\sum_{r=-\infty}^{\infty} c_{2r}e^{-2rix} +$$
$$+ \frac{b}{\delta}\left[e^{-\mu x}\sum_{r=-\infty}^{\infty} c_{2r}e^{-2rix}\sum_{r=-\infty}^{\infty}\frac{c_{2r}}{2ri+pi+\mu}e^{(2ri+pi+\mu)x} - \right.$$
$$\left. - e^{\mu x}\sum_{r=-\infty}^{\infty} c_{2r}e^{2rix}\sum_{r=-\infty}^{\infty}\frac{c_{2r}}{-2ri+pi-\mu}e^{(-2ri+pi-\mu)x}\right].$$

The first terms on the right-hand side containing the arbitrary constants A and B constitute the general solution of a homogeneous Mathieu equation.

Of interest in this case is the particular solution which we write as /67

$$y=\text{particular}=\frac{b}{\delta}\Big[\sum_{r=-\infty}^{\infty}c_{2r}e^{-2ri x}\sum_{r=-\infty}^{\infty}\frac{c_{2r}}{2ri+pi\mu}e^{(2ri+pi)x}- \\ -\sum_{r=-\infty}^{\infty}c_{2r}e^{2ri x}\sum_{r=-\infty}^{\infty}\frac{c_{2r}}{-2ri+pi-\mu}e^{(-2ri+ip-\mu)x}\Big]. \tag{2.37}$$

From this expression it follows that a particular solution of a nonhomogeneous Mathieu equation remains finite when $x \to \infty$ if

$$(2ri+pi\pm\mu)\neq 0,$$

where r = ... -2, -1, 0, 1, 2, ... is any integer.

We readily see that, when (2ri + pi $\pm$ μ) $\to$ 0, we have

$$\frac{c_{2r}}{2ri+pi-\mu}\to\infty,$$

or

$$\frac{c_{2r}}{2ri+pi+\mu}\to\infty.$$

Hence the resonance condition for the case of a nonhomogeneous Mathieu equation is expressed by the relation

$$2ri+pi\pm\mu=0. \tag{2.38}$$

§ 2.6. Differences (see [10, 25]).

We let y = f(x) be a function of the variable x which, within a given interval of the variation of x, assumes finite values. The variable x may vary continuously or may assume only separate values, e.g., 0, 1, 2, 3, ... , n, ... Within that interval we select any (n + 1) successive equidistant points x, x + Δx, x + 2Δx, ..., x + nΔx, designating as y_x, $y_{x+\Delta x}$, $y_{x+2\Delta x}$,..., $y_{x+n\Delta x}$ the corresponding values of the function at those points. The expressions

$$\left.\begin{aligned} y_{x+\Delta x} - y_x &= \Delta y_x, \\ y_{x+2\Delta x} - y_{x+\Delta x} &= \Delta y_{x+\Delta x}, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ y_{x+n\Delta x} - y_{x+(n-1)\Delta x} &= \Delta y_{x+(n-1)\Delta x} \end{aligned}\right\} \qquad (2.39)$$

are called first-order finite differences of the function f(x) at the points x, x + Δx, ... , x + nΔx. /68

The first-order finite differences serve to form the second-order finite differences:

$$\left.\begin{aligned} \Delta y_{x+\Delta x} - \Delta y_x &= \Delta^2 y_x, \\ \Delta y_{x+2\Delta x} - \Delta y_{x+\Delta x} &= \Delta^2 y_{x+\Delta x}, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \Delta y_{x+(n-1)\Delta x} - \Delta y_{x+(n-2)\Delta x} &= \Delta^2 y_{x+(n-2)\Delta x}. \end{aligned}\right\} \qquad (2.40)$$

Similarly, the third-order finite differences will be:

$$\left.\begin{aligned} \Delta^2 y_{x+\Delta x} - \Delta^2 y_x &= \Delta^3 y_x, \\ \Delta^2 y_{x+2\Delta x} - \Delta^2 y_{x+\Delta x} &= \Delta^3 y_{x+\Delta x}, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \Delta^2 y_{x+(n-2)\Delta x} - \Delta^2 y_{x+(n-3)\Delta x} &= \Delta^3 y_{x+(n-3)\Delta x}. \end{aligned}\right\} \qquad (2.41)$$

And an n-th-order finite difference will always be:

$$\Delta^{n-1} y_{x+\Delta x} - \Delta^{n-1} y_x = \Delta^n y_x.$$

To form the n-th-order finite difference, we had to know n + 1 successive values of the functions y_x, $y_{x+\Delta x}$, $y_{x+n\Delta x}$. By using the equalities (2.39), (2.40) and (2.41) we can express the finite differences as successive values of the functions:

$$\begin{aligned} \Delta^2 y_x &= y_{x+2\Delta x} - 2y_{x+\Delta x} + y_x, \\ \Delta^3 y_x &= y_{x+3\Delta x} - 3y_{x+2\Delta x} + 3y_{x+\Delta y} - y_x, \\ \Delta^n y_x &= y_{x+n\Delta x} - n y_{x+(n-1)\Delta x} + \\ &+ \frac{n(n-1)}{2\cdot 1} y_{x+(n-2)\Delta x} - \frac{n(n-1)(n-2)}{3\cdot 2\cdot 1} y_{x+(n-3)\Delta x} + \\ &+ \frac{n(n-1)(n-2)(n-3)}{4\cdot 3\cdot 2\cdot 1} y_{x+(n-4)\Delta x} - \cdots + (-1)^n y_x. \end{aligned}$$

Here the coefficients n, $\frac{n(n-1)}{2\cdot 1}$, $\frac{n(n-1)(n-2)}{3\cdot 2\cdot 1}$, ... are binomial coefficients and are written for short as

$$\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}.$$

respectively. If the variable x assumes only the integral values x = 0, 1, 2, /69
... , n, while $\Delta x = 1$, the n-th-order finite difference will be written as

$$\Delta^n y_x = y_{x+n} - \binom{n}{1} y_{x+n-1} + \binom{n}{2} y_{x+n-2} - \binom{n}{3} y_{x+(n-3)} + \ldots + (-1)^n \binom{n}{n} y_x. \tag{2.42}$$

The separate values of the function y_{x+1}, y_{x+2}, ... , y_{x+n} can in turn be expressed in terms of y_x and the successive values of the fine differences Δy_x, $\Delta^2 y_x$ ... $\Delta^n y_x$:

$$\left.\begin{aligned} &y_{x+1} = y_x + \Delta y_x, \\ &y_{x+2} = y_{x+1} + \Delta y_{x+1} = y_x + 2\Delta y_x + \Delta^2 y_x, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &y_{x+n} = y_x + \binom{n}{1} \Delta y_x + \binom{n}{2} \Delta^2 y_x + \ldots + \binom{n}{n} \Delta^n y_x. \end{aligned}\right\} \tag{2.43}$$

§ 2.7. Ordinary Linear Difference Equations.

A difference equation is an equation of the type

$$F(x, y_x, \Delta y_x, \Delta^2 y_x, \ldots, \Delta^n y_x) = 0, \tag{2.44}$$

where F is a given function, y the function sought.

When the differences are replaced by successive values of the functions according to formula (2.42), equation (2.44) becomes

$$\Phi(x, y_x, y_{x+1}, \ldots, y_{x+n}) = 0.$$

If the left-hand side of this equation is a linear function of y_x, y_{x+1}, ... , y_{x+n}, we can then write it as

$$p_x^0 y_x + p_x^1 y_{x+1} + \ldots + p_x^n y_{x+n} = u_x, \tag{2.45}$$

where p_x and u_x are either functions of x or given numbers.

Equation (2.45) is called a linear difference equation. If its right-hand side $u_x = 0$, it is called a linear homogeneous difference equation. If, in

addition, all the p_x are constant numbers (i.e., do not depend on x), it is called a linear homogeneous difference equation with constant coefficients. /70

We write it in the form

$$y_{x+n} + a_1 y_{x+n-1} + \ldots + a_n y_x = 0. \tag{2.46}$$

Here a_1, a_2, ... , a_n are given numbers.

Henceforward we shall be interested only in equations of the form (2.46). If the equation contains both y_{x+n} and y_x, it is called an equation of n-th order. If it does not contain y_x but does contain, let us say, y_{x+1}, then replacing the independent variable x + 1 with x lowers the order of the equation to unity. The equation

$$a_1 y_{x+3} + a_2 y_{x+2} + a_3 y_{x+1} + a_4 y_x = 0$$

is an equation of third order, for it contains both y_{x+3} and y_x. If $a_1 = 0$ (i.e., if there is no y_{x+3}), then $a_2 y_{x+2} + a_3 y_{x+1} + a_4 y_x = 0$ will be an equation of second order. We now assume that it does not contain y_x ($a_4 = 0$), whereupon

$$a_1 y_{x+3} + a_2 y_{x+2} + a_3 y_{x+1} = 0$$

will also be an equation of second order. Replacing x + 1 with x changes it to

$$a_1 y_{x+2} + a_2 y_{x+1} + a_3 y_x = 0.$$

We return now to equation (2.46). Its solution is a function y_x which makes it left-hand side identically equal to zero (i.e., independently of x), and is usually sought in the form $y_x = \beta^x$, where β is the number to be determined.

Substituting the assumed solution in (2.46) and canceling out β^x, we get

$$\beta^n + a_1 \beta^{n-1} + a_2 \beta^{n-2} + \ldots + a_n = 0, \tag{2.47}$$

which is called the characteristic equation for the difference equation (2.46). The characteristic equation has roots different from zero, since $a_n \neq 0$. Naturally, the roots of (2.47) can be both simple and multiple. We shall assume them all simple. We then get n different particular solutions of (2.46) in the form /71

$$\left.\begin{array}{ll} \text{when } x = 0 & y_x = 0, \\ \text{when } x = 1 & y_x = 1. \end{array}\right\} \tag{2.50}$$

Then $C_1 = \frac{1}{6}$, $C_2 = -\frac{1}{6}$. Under these conditions the solution will be

$$y_x = \frac{1}{6}(-1)^x - \frac{1}{6}(-7)^x. \tag{2.51}$$

Using this same example, we shall show another way of solving homogeneous difference equations having constant coefficients with the condition that x assume only the integral values x = 0, 1, 2, ... We define the interval of the variation of x as, let us say, $0 \leq x \leq 5$. Under these conditions the difference equations can be regarded as a recursion formula enabling us to calculate successively from the initial conditions the value of the function for x = 0, 1, 2, 3, 4, 5. When x = 0, $y_2 + 8y_1 + 7y_0 = 0$. Accordingly,

$$\begin{array}{lll} \text{when} & x=1 & y_3+8y_2+7y_1=0, \\ \text{when} & x=2 & y_4+8y_3+7y_2=0, \\ \text{when} & x=3 & y_5+8y_4+7y_3=0, \\ \text{when} & x=4 & y_6+8y_5+7y_4=0, \\ \text{when} & x=5 & y_7+8y_6+7y_5=0. \end{array}$$

Taking into account the initial conditions (2.50), we find $y_2 = -8$, $y_3 = 57$, $y_4 = -400$, $y_5 = 2{,}801$, $y_6 = -19{,}608$, $y_7 = 137{,}257$.

As we see, in order to calculate the value of y_7 by this method, we had to calculate $y_1, y_2, \ldots, y_6$. We calculate now the desired value of the function directly from the solution (2.51). To do so, we substitute in it x = 7, whereupon /73

$$y_7 = \frac{1}{6}(-1)^7 - \frac{1}{6}(-7)^7 = 137\,257.$$

This value, of course, coincides with that of the function for x = 7 calculated above. When we consider that, as the interval of the variation of x widens, the number of algebraic equations needed to calculate the function increases, the advantage of the difference-equation device becomes obvious.

In our study of mechanisms with elastic couplings it will not be necessary to seek general solutions of difference equations. We are interested only in the roots of a characteristic equation of the (2.47) type. So, when we come to analyze difference-equation systems, we shall confine ourselves to setting up the characteristic equation.

$$y_x^{(\nu)} = \beta_\nu^x \quad (\nu = 1, 2, \ldots, n).$$

In this case the general solution of (2.46) will be

$$y_x = C_1\beta_1^x + C_2\beta_2^x + \ldots + C_n\beta_n^x. \tag{2.48}$$

Here C_1, C_2, ... , C_n are n arbitrary constants, determinable from the n initial conditions. If some of the characteristic equation's roots are complex, those are always conjugate pairs (since the coefficients of the characteristic equation are real).

We shall assume, for simplicity's sake, that of the n roots of the characteristic equation (2.47) two are complex:

$$\beta_1 = \varrho(\cos\varphi + i\sin\varphi),$$
$$\beta_2 = \varrho(\cos\varphi - i\sin\varphi).$$

In this case $\rho^x\cos\phi_x$ and $\rho^x\sin\phi x$ are particular solutions of (2.46), and the general solution is written as

$$y_x = C_1\varrho^x\cos\varphi x + C_2\varrho^x\sin\varphi x + C_3\beta_3^x + \ldots + C_n\beta_n^x.$$

Now if $\beta_1 = \beta_2 = \beta_s$ is the s-multiple root of the characteristic equation, the particular solutions of (2.46) will be β_1^x, $x\beta_1^x$, $x^2\beta_1^x$, ..., $x^{s-1}\beta_1^x$, and the general solution will be

$$y_x = (C_1 + C_2x + \ldots + C_sx^{s-1})\beta_1^x + C_{s+1}\beta_{s+1}^x + \ldots + C_n\beta_n^x.$$

Let us take an example. The equation

$$y_{x+2} + 8y_{x+1} + 7y_x = 0. \tag{2.49}$$

being assumed given, we seek its solution in the form $y_x = \beta^x$. The characteristic equation will be $\beta^2 + 8\beta + 7 = 0$, its roots $\beta_1 = 1$, $\beta_2 = -7$. The general solution is written as /72

$$y_x = C_1(-1)^x + C_2(-7)^x.$$

We set the initial conditions, stipulating that

§ 2.8. A System of Difference Equations with Constant Coefficients.

If y_x, z_x, u_x ... are functions of the variable x, the system of equations

$$\left.\begin{aligned}
&\sum_{i=0}^{k} a_i y_{x+i} + \sum_{i=0}^{l} b_i z_{x+i} + \sum_{i=0}^{m} c_i u_{x+i} + \ldots = 0,\\
&\sum_{i=0}^{k'} a'_i y_{x+i} + \sum_{i=0}^{l'} b'_i z_{x+i} + \sum_{i=0}^{m'} c'_i u_{x+i} + \ldots = 0,\\
&\sum_{i=0}^{k''} a''_i y_{x+i} + \sum_{i=0}^{l''} b''_i z_{x+i} + \sum_{i=0}^{m''} c''_i u_{x+i} + \ldots = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \qquad (2.52)$$

in which a, b, c are constants, is called a homogeneous system of difference equations with constant coefficients. The number of equations should equal the number of unknown functions. By eliminating the unknown functions, we can get one equation containing, for instance, only the one unknown function y_x. The order of this equation will obviously not exceed $r = \overline{k} + \overline{l} + \overline{m} + \ldots$ if $\overline{k}$, $\overline{l}$, $\overline{m}$, ... are the largest of the numbers k, k' ..., l, l',..., m, m', ... etc. The equation for y_x is solved by the method described in the preceding section. Its general solution contains r arbitrary constants. With the aid of the already known function y_x we determine the remaining unknown functions z_x, u_x, ..., which are linearly dependent on it. /74

Let us take an example. We assume the system

$$\left.\begin{aligned}
y_{x+2} + 4y_x + z_{x+1} - z_x = 0,\\
y_{x+1} - y_x - z_{x+1} - z_x = 0.
\end{aligned}\right\}$$

to be given. Here k = 2, k' = 1, l = 1, l' = 1, whence $r \leq 3$. Solving these equations for y_{x+2} and z_{x+1}, we get

$$y_{x+2} = -4y_x - z_{x+1} + z_x, \qquad (2.53)$$

$$z_{x+1} = y_{x+1} - y_x - z_x. \qquad (2.54)$$

Eliminating z_x from them, we find

$$y_{x+2} = y_{x+1} - 5y_x - 2z_{x+1}. \qquad (2.55)$$

Substituting z_{x+1} from (2.54) in (2.71), we get

$$y_{x+2} = -y_{x+1} - 3y_x + 2z_x.$$

After replacement of x with x + 1 this equation will be

$$y_{x+3} = -y_{x+2} - 3y_{x+1} + 2z_{x+1}. \tag{2.56}$$

Eliminating z_{x+1} from (2.55) and (2.56), we obtain

$$y_{x+3} + 2y_{x+2} + 2y_{x+1} + 5y_x = 0.$$

This equation contains only the one unknown function y_x. It is of third order and is solved by the method already known. The characteristic equation for it will be

$$\beta^3 + 2\beta^2 + 2\beta + 5 = 0.$$

After determining y_x, we can easily determine z_x:

$$z_x = \frac{1}{2}(y_{x+2} + y_{x+1} + 3y_x).$$

We shall now demonstrate another method for determining the unknown functions directly from a given system of homogeneous difference equations with constant coefficients. We seek solutions of system (2.52) in the form /75

$$y_x = D\beta^x, \qquad z_x = E\beta^x, \qquad u_x = F\beta^x, \ldots \tag{2.57}$$

After substituting these expression in (2.52), we get the following system of linear homogeneous equations for determining the coefficients D, E, F, ...

$$\left.\begin{aligned} D\sum_{i=0}^{k} a_i\beta^i + E\sum_{i=0}^{l} b_i\beta^i + F\sum_{i=0}^{m} c_i\beta^i + \ldots = 0, \\ D\sum_{i=0}^{k'} a'_i\beta^i + E\sum_{i=0}^{l'} b'_i\beta^i + F\sum_{i=0}^{m'} c'_i\beta^i + \ldots = 0, \\ D\sum_{i=0}^{k''} a''_i\beta^i + E\sum_{i=0}^{l''} b''_i\beta^i + F\sum_{i=0}^{m''} c''_i\beta^i + \ldots = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{2.58}$$

This system of equations has roots D, E, F, ... different from zero only in the case in which the determinant

$$\Delta=\begin{vmatrix} \sum\limits_{i=0}^{k} a_i\beta^i & \sum\limits_{i=0}^{l} b_i\beta^i & \sum\limits_{i=0}^{m} c_i\beta^i \dots \\ \sum\limits_{i=0}^{k'} a'_i\beta^i & \sum\limits_{i=0}^{l'} b'_i\beta^i & \sum\limits_{i=0}^{m'} c'_i\beta^i \dots \\ \sum\limits_{i=0}^{k''} a''_i\beta^i & \sum\limits_{i=0}^{l''} b''_i\beta^i & \sum\limits_{i=0}^{m''} c''_i\beta^i \dots \\ \dots\dots\dots\dots\dots \end{vmatrix}=0. \tag{2.59}$$

The condition $\Delta = 0$, constituting the characteristic equation, gives for β an equation of order r, where r, as before, is less than or equal to the sum $\overline{k} + \overline{l} + \overline{m} + \dots$ After determining the roots β of the characteristic equation, we easily find the general solution of system (2.52).

Returning to our previous example, we solve the system of difference equations given with it by the method just explained. We seek its solution in the form $y_x = D\beta^x$, $z_x = E\beta^x$. Then we have /76

$$D\beta^{x+2}+4D\beta^x+E\beta^{x+1}-E\beta^x=0,$$
$$D\beta^{x+1}-D\beta^x-E\beta^{x+1}+E\beta^x=0.$$

Canceling out β^x, we get

$$D\beta^2+4D+E\beta-E=0,$$
$$D\beta-D-E\beta-E=0.$$

This system's characteristic equation will be

$$\begin{vmatrix} \beta^2+4 & \beta-1 \\ \beta-1 & -(\beta+1) \end{vmatrix}=0,$$

or, in its expanded form, we get the equation that we already know

$$\beta^3+2\beta^2+2\beta+5=0.$$

Even this simple example suffices to demonstrate the advantage of the second way of solving a system of difference equations having constant coefficients. In solving more complicated systems, the function-elimination method is so unwieldly as to be virtually impracticable.

§ 2.9. Schur Inequalities (see [3, 57]).

To analyze the stability of the periodic states of motion of vibroimpact

systems, we have to evaluate the roots of the characteristic equation. If in absolute value all the roots of that equation are less than unity, the motion is stable. We cite here one of the theorems of higher algebra, known as the Schur theorem, according to which the conditions necessary and sufficient for all the roots of the polynomial

$$f(z) = a_0\beta^n + a_1\beta^{n-1} + \ldots + a_n \tag{2.60}$$

to lie within a unit circle are the inequalities

$$|w_i| < 1 \quad (i = 0, 1, \ldots, n-1).$$

We adopt this theorem without proof, because for us it has a merely subsidiary importance, whereas we shall formally construct the functions ω_i that the theorem is about. Along with (2.60), we consider another polynomial /77

$$f^*(\beta) = a_n\beta^n + a_{n-1}\beta^{n-1} + \ldots + a_0 \tag{2.61}$$

and form the polynomial

$$f_1(\beta) = \frac{1}{\beta}[a_0 f(\beta) - a_n f^*(\beta)] = a_0^{(1)}\beta^{n-1} + \ldots + a_{n-1}^{(1)}.$$

After that, we calculate

$$f_2(\beta) = \frac{1}{\beta}[a_0^{(1)} f_1(\beta) - a_{n-1}^{(1)} f_1^*(\beta)] = a_0^{(2)}\beta^{n-2} + \ldots + a_{n-2}^{(2)},$$

$$f_3(\beta) = \frac{1}{\beta}[a_0^{(2)} f_2(\beta) - a_{n-2}^{(2)} f_2^*(\beta)] = a_0^{(3)}\beta^{n-3} + \ldots + a_{n-3}^{(3)},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f_k(\beta) = \frac{1}{\beta}[a_0^{(k-1)} f_{k-1}(\beta) - a_{n-k+1}^{(k-1)} f_{k-1}^*(\beta)] =$$

$$= a_0^{(k)}\beta^{n-1} + \ldots + a_{n-k}^{(k)}$$

$$(k = 1, 2, 3, \ldots, n);$$

whereupon

$$w_0 = \frac{a_n}{a_0}, \quad w_1 = \frac{a_{n-1}^{(1)}}{a_0^{(1)}}, \quad w_2 = \frac{a_{n-2}^{(2)}}{a_0^{(2)}}, \quad w_{n-1} = \frac{a_1^{(n-1)}}{a_0^{(n-1)}}. \tag{2.62}$$

As examples, we set up the functions ω_i for the second- and third-order equations.

Assuming $f(\beta) = a_0\beta^2 + a_1\beta + a_2$, we have

$$f^*(\beta) = a_2\beta^2 + a_1\beta + a_0,$$

$$f_1(z) = \frac{1}{\beta}[a_0(a_0\beta^2 + a_1\beta + a_2) - a_2(a_2\beta^2 + a_1\beta + a_0)] =$$

$$= \frac{1}{\beta}[(a_0^2 - a_2^2)\beta^2 + (a_0a_1 - a_2a_1)\beta] =$$

$$= (a_0^2 - a_2^2)\beta + (a_0a_1 - a_2a_1) = a_0^{(1)}\beta + a_1^{(1)}.$$

$$w_0 = \frac{a_2}{a_0}, \quad w_1 = \frac{a_1^1}{a_0^{(1)}} = \frac{a_1}{a_0 + a_2}.$$

Now assuming

$$f(\beta) = a_0\beta^3 + a_1\beta^2 + a_2\beta + a_3.$$

we get /78

$$f^*(\beta) = a_3\beta^3 + a_2\beta^2 + a_1\beta + a_0,$$

$$f_1(\beta) = \frac{1}{\beta}[a_0(a_0\beta^3 + a_1\beta^2 + a_2\beta + a_3) -$$

$$- a_3(a_3\beta^3 + a_2\beta^2 + a_1\beta + a_0)] = \frac{1}{\beta}[(a_0^2 - a_3^2)\beta^3 +$$

$$+ (a_0a_1 - a_3a_2)\beta^2 + (a_0a_2 - a_3a_1)\beta] = (a_0^2 - a_3^2)\beta^2 +$$

$$+ (a_0a_1 - a_3a_2)\beta + a_0a_2 - a_3a_1 = a_0^{(1)}\beta^2 + a_1^{(1)}\beta + a_2^{(1)}.$$

Here

$$a_0^{(1)} = a_0^2 - a_3^2, \quad a_1^{(1)} = a_0a_1 - a_3a_2, \quad a_2^{(1)} = a_0a_2 - a_3a_1,$$

$$f_2(\beta) = \frac{1}{\beta}[a_0^{(1)}(a_0^{(1)}\beta^2 + a_1^{(1)}\beta + a_2^{(1)}) - a_2^{(1)}(a_2^{(1)}\beta^2 + a_1\beta^{(1)} + a_0^{(1)})] =$$

$$= \frac{1}{\beta}[(a_0^{(1)2} - a_2^{(1)2})\beta^2 + (a_0^{(1)}a_1^{(1)} - a_2^{(1)}a_1^{(1)})\beta] =$$

$$= (a_0^{(1)2} - a_2^{(1)2})\beta + a_0^{(1)}a_1^{(1)} - a_2^{(1)}a_1^{(1)} = a_0^{(2)}\beta + a_1^{(2)},$$

$$a_0^{(2)} = a_0^{(1)2} - a_2^{(1)2} = (a_0^2 - a_3^2)^2 - (a_0a_2 - a_3a_1)^2,$$

$$a_1^{(2)} = a_0^{(1)}a_1^{(1)} - a_2^{(1)}a_1^{(1)} = (a_0^2 - a_3^2)(a_0a_1 - a_3a_2) -$$

$$- (a_0a_2 - a_3a_1)(a_0a_1 - a_3a_2).$$

Now

$$w_0 = \frac{a_3}{a_0}, \quad w_1 = \frac{a_2^{(1)}}{a_0^{(1)}} = \frac{a_0a_2 - a_3a_1}{a_0^2 - a_3^2},$$

$$w_2 = \frac{a_1^{(2)}}{a_0^{(2)}} = \frac{a_0a_1 - a_3a_2}{a_0(a_0 + a_2) - a_3(a_3 + a_1)}.$$

The information on difference equation and Schur inequalities provided in the last three sections of this chapter will be needed when we analyze the stability of the motion of vibroimpact systems in Chapters, 7, 8 and 9.

CHAPTER 3

MECHANISMS WITH ELASTIC COUPLINGS

§ 3.1. Elastic Elements. /79

The basic part of an elastic link or coupling is the elastic element. The main characteristic of that element is the functional relation expressing the magnitude of a force applied to it as a function of its deformation.

In a very simple case, e.g., the stretching of an elastic bar by a central force (Figure 3.1), this characteristic is expressed directly by Hooke's well-known law, according to which

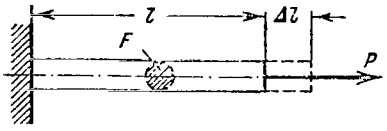

Figure 3.1.

$$\sigma = \frac{P}{F} = E\varepsilon, \tag{3.1}$$

where σ is the stress (kg/cm^2), F the cross-sectional area (cm^2), E the elasticity modulus (kg/cm^2), and $\varepsilon = \frac{\Delta l}{l}$ the unit elongation.

Here it is assumed that the stresses are the same at all points of the bar's cross-section.

We rewrite formula (3.1) as follows:

$$P = \frac{EF}{l}\Delta l = a\Delta l. \tag{3.2}$$

As we see, Hooke's law gives us a linear relation between the force applied to the bar and its resulting deformation. Actually, this relation is approximate, agreeing fairly well with the true stress-strain diagram only /80 initially, i.e., until the stress σ reaches a certain proportionality limit, which is different for different materials.

Elastic elements designed for pure tension are seldom used in machines and instruments. Bending and torsional strains are more common. As an example, we shall determine the elastic characteristic of a flat bending spring (Figure 3.2,a). Figure 3.2,b shows a portion of it enlarged. In the bending of an elastic element the stresses in its cross-section are distributed unevenly. They increase outwardly from the middle to the periphery, so that on one side of the so-called neutral layer the spring's material is under tension, on the other side under compression. If we assume that the stress on the material's fibers is proportional to their elongation (Hooke's law), and that the elongation Δ is proportional to the distance e from the neutral layer, then, in accordance with Figure 3.2,b, we can write

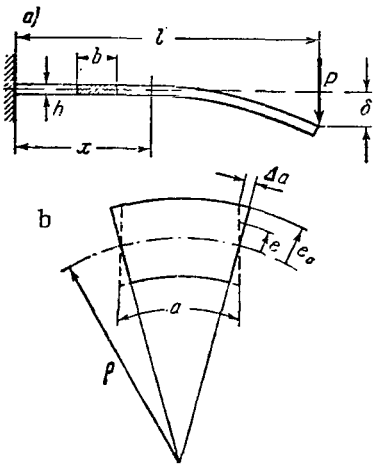

Figure 3.2.

$$\frac{\sigma}{\sigma_a}=\frac{e}{e_a}. \tag{3.3}$$

Designating the cross-sectional area of the elementary fiber as df, we get the magnitude of the elementary tensile force dP:

$$dP = \sigma df.$$

The combined moment of the elementary forces/81 acting upon the cross-section is

$$M=\int_F e\,dP=\int_F \sigma e\,df.$$

Substituting in this the value of σ from (3.3), we find

$$M=\frac{\sigma_a}{e_a}\int_F e^2\,df.$$

We must integrate over the entire area of the cross-section, which we indicate by the letter F under the integral sign.

The expression

$$J=\int_F e^2\,df$$

is called the cross-section's axial moment of inertia. For determining the stresses we frequently use not this expression but the so-called moment of resistance to bending

$$W=\frac{J}{e_a}. \tag{3.4}$$

Hence the bending moment acting in a certain section x produces in the section's extreme fibers the stress

$$\sigma_a=\frac{M_x}{W}. \tag{3.5}$$

With the type of fastening and elastic-element stress that we are considering, the bending moment in the section x is

$$M_x = P(l - x).$$

It reaches its maximum value when x = 0. Then

$$\sigma_a = \frac{Pl}{W} . \tag{3.6}$$

An elastic element is designed so that σ_a will be clearly below the elastic limit, i.e., less than that apparent stress at which we see the first signs of a plastic strain remaining after the stress has been removed. /82

Now we find the deflection of our flat spring when acted upon by a force P. To do this, we must set up an equation for the elastic curve of the bent spring. From the similarity of the triangles in Figure 3.2,b, it follows that

$$\frac{\varrho}{a} = \frac{e_a}{\Delta a} . \tag{3.7}$$

Here ρ is the radius of curvature of the spring's neutral layer in the section in question. The relation

$$\frac{\Delta a}{a} = \varepsilon$$

defines the unit elongation of the spring material. Hence

$$\varrho = \frac{e_a}{\varepsilon} .$$

Substituting in this the expression of ε from (3.1), we get

$$\varrho = \frac{e_a E}{\sigma_a} .$$

Determining from (3.4) and (3.5) the ratio e_a/σ_a, we find that the curvature of the spring's elastic curve at the point in question is

$$K = \frac{1}{\varrho} = \frac{M}{EJ} = \frac{P(l-x)}{EJ} . \tag{3.8}$$

From analytic geometry we know that the curvature of a curve $y = y(x)$ is expressed by the formula:

$$K = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} . \tag{3.9}$$

Disregarding dy/dx, which is small compared to unity (by virtue of the small curvature), we get an approximative formula for determining the curvature and a correspondingly approximative differential equation for the elastic curve

$$\frac{d^2y}{dx^2} = \frac{P}{EJ}(l - x). \tag{3.10}$$

Integrating it twice and determining the arbitrary integration constants from the initial conditions y = 0, dy/dx = 0 when x = 0, we get the following equation from the elastic curve: /83

$$y = \frac{P}{EJ} \cdot \frac{x^2}{6}(3l - x).$$

To plot the elastic element's characteristic, we have to find the deflection $y = \delta$ at the point of application of the force, i.e., for $x = l$. We get

$$\delta = \frac{Pl^3}{3EJ},$$

whence we find the sought-for relation $P = P(\delta)$ between the force applied to the elastic element and its deformation

$$P(\delta) = k\delta. \tag{3.11}$$

The constant $k = \frac{3EJ}{l^3}$ is called the elastic element's stiffness coefficient or the spring constant. Since the deflection is expressed in units of length, the magnitude of the stiffness coefficient will be expressed in terms of [k] = kg/cm.

The characteristic (3.11) is plotted in Figure 3.3, its slope depending on the stiffness of the elastic element, so that

$$\tan \alpha = k.$$

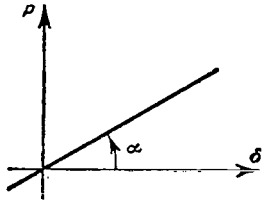

Figure 3.3.

As we see, this characteristic is linear. But once we have carried through in detail the intermediate calculations, recalling the assumptions that we made in deriving equation (3.11), it becomes clear that this linear law only approximately reflects the actual relationship between the force applied to an elastic element and its deformation. And we have not yet even allowed for the fact that the actual conditions of an elastic element's constraint, being very often far from ideal, may themselves substantially affect the characteristic of the elastic coupling as a whole.

This is why, for designing and calculating mechanisms with elastic couplings, it is necessary in important cases to calibrate the elastic element, i.e., to determine their elastic characteristics experimentally. /84

In some cases it is necessary that an elastic element have a nonlinear characteristic. Various design solutions make it possible to obtain such characteristics with the use of ordinary linear elements. Figure 3.4,a is an example. There, as the deflection of the spring 1 increases, an even longer segment of it is in contact with the support 2. The spring's effective length thus decreases, whence its stiffness coefficient increases. Figure 3.4,b is a plot of the characteristic for that design. Such a characteristic is usually called stiff.

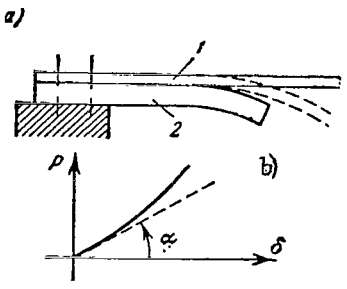

Figure 3.4.

When an elastic element undergoes deformation, a potential energy Π of elastic deformation accumulates within it. In the process of its return to the undeformed state this energy is expended. The elastic forces perform an amount of work dependent solely on the elastic element's initial and final state and not on the process of transition from the one state to the other.

Thus the work done by the elastic forces in producing an elementary displacement equals P dδ. The work done by the elastic element in returning to its undeformed state is equal to its potential energy in the deformed position, and is defined as:

$$\Pi(\delta) = -\int_{\delta}^{0} P\,d\delta.$$

The minus sign appears here because, as the elastic element returns to its undeformed state, the force applied to it is directed against the displacement. /85

For a flat spring with a linear characteristic we get

$$\Pi(\delta) = -\int_{\delta}^{0} k\delta\,d\delta = \frac{k\delta^2}{2}. \qquad (3.12)$$

Generally, the potential energy of deformation of any elastic element having a linear characteristic is a square-law function of the quantity characterizing the deformation.

In the mechanisms of machines and instruments flat springs are used under different load conditions and with fastenings of varying design. The stiffness coefficients for several cases are given in Figure 3.5 as an example.

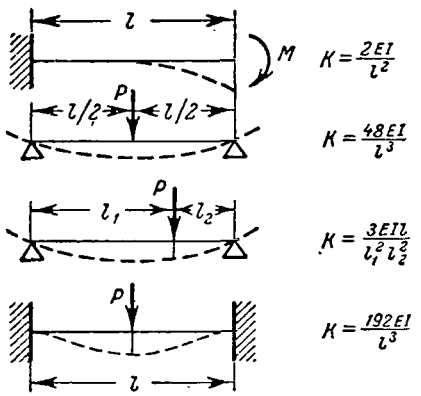

Figure 3.5.

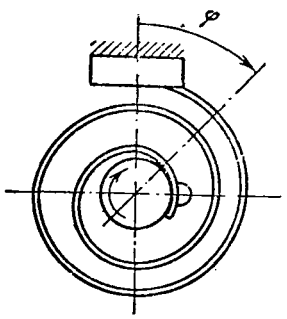

Figure 3.6.

§ 3.2. Spiral and Helical Springs

Compared to flat springs, spiral and helical springs possess the double advantage of assuring larger elastic displacements while having relatively small over-all dimensions.

Spiral springs (Figure 3.6) are normally made of material in the form of a strip or band. Like flat springs, they are of the bending type, but are usually connected to links that perform a swinging or rotary motion. Their deformations are thus measured by the angles of rotation of the respective elements of the kinematic pair, the load upon them by the moment of rotation, whose relation to the deformation is expressed by the characteristic: /86

$$M = q\phi \quad [q] = \text{kg cm/rad}. \tag{3.13}$$

The stiffness of a spiral spring is defined by the formula:

$$q = \frac{EJ}{l}, \tag{3.14}$$

where l is the length of the spring's elastic element in the load-free state, which is calculated approximately from the obvious relation

$$l = (r_{ext} + r_{int})\pi n,$$

where r_{ext} is the spring's external radius, r_{int} its internal radius, n being the number of coils.

Along with flat and spiral springs, helical springs too may be of the bending type. One example of the design of a kinematic pair with an elastic element of this kind is shown in Figure 3.7. The characteristic and stiffness of the element are determined with the relations (3.13) and (3.14).

By analogy with (3.12), the potential energy of elements such as spiral or helical bending springs is written as /87

$$\Pi(\varphi) = \frac{q\varphi^2}{2}.$$

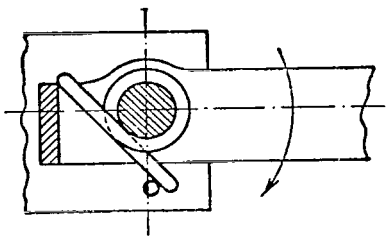

Figure 3.7.

But most often the helical springs in the mechanisms of machines and instruments are tension or compression springs. Then the spring material (usually round wire) is subjected to torsion.

Let us recall the fundamental premises and assumptions on the basis of which we determine the elastic characteristics and constants of elastic elements of this type. When a round bar one of whose ends is immovably fastened (Figure 3.8) is subjected to torsion, there arise in the bar material the so-called tangential stresses τ, which lie in its cross-sectional plane. These produce shear strains whereby adjacent sections of the bar become deformed to an extent characterized by the angle γ. According to Hooke's law, the magnitude of the tangential stress is tied to the magnitude of the strain by the linear relation

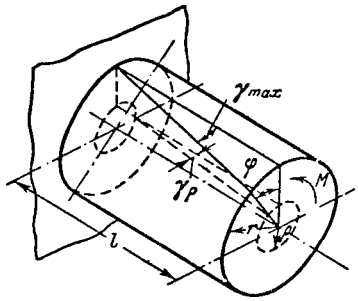

Figure 3.8.

$$\tau = G \cdot \gamma, \tag{3.15}$$

where G is the modulus of elasticity in shear.

The stress reaches its maximum $\tau = \tau_{max}$ on the bar's surface. If we assume, moreover, that the strains and stresses vary in proportion to the distance from the bar's axis, we then have

$$\frac{\gamma_\rho}{\gamma_{max}} = \frac{\tau_\rho}{\tau_{max}} = \frac{\rho}{r}. \tag{3.16}$$

To find the relation between the moment of rotation applied to the bar and the stress produced in it, we proceed as follows. In the cross-sectional plane we select the elementary area df at a distance ρ from the axis. The force /88 equivalent to the stress acting upon this area is

$$dP = \tau_\rho \, df.$$

Summing these quantities for the entire cross-section, we get the moment of the internal forces, which counterbalances the moment of the external forces applied to the bar:

$$M = \int_F \tau_\rho \cdot \rho \, df.$$

Substituting in this the expression of τ_ρ from (3.16), we get

$$M = \frac{\tau_{max}}{r} \int_F \varrho^2 \, df.$$

The quantity $J_p = \int_F \rho^2 \, df$ is called the section's polar moment of inertia.

Hence

$$M = \frac{\tau_{max}}{r} J_p = \tau_{max} W_p, \tag{3.17}$$

where W_p is the sectional modulus of torsion.

For a circular cross-section

$$J_p = \frac{\pi d^4}{32}, \qquad W_p = \frac{\pi d^3}{16}.$$

When a bar is subjected to torsion by a moment M, a cross-section of the bar at a distance l from the fastening will rotate through an angle ϕ (Figure 3.8). Hence a cross-section at a distance equal to unity will rotate through an angle ϕ/l. Its magnitude will be

$$\frac{\varphi}{l} = \frac{\gamma_{max}}{r}.$$

Taking into account (3.15), we get

$$\varphi = \frac{\tau_{max} l}{Gr}. \tag{3.18}$$

According to (3.17), the ratio $\tau_{max}/r = M/J_p$, whence the characteristic of that elastic element can be written as /89

$$M = q\phi, \tag{3.19}$$

where the stiffness is

$$q = \frac{GJ_p}{l}.$$

A helical spring whose coil diameter is much larger than the diameter of the wire comprising the spring material (Figure 3.9) is calculated with similar formulae. The relation between a force P applied to a spiral spring and the latter's resulting linear deformation is

$$P = kx. \tag{3.20}$$

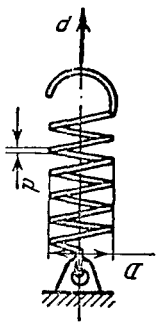

Figure 3.9.

The spring's stiffness is defined by the formula

$$k = \frac{Gd^4}{8nD^3},$$

where n is the number of coils.

If in the system of a mechanism with elastic couplings there are two elastic elements connected in parallel (Figure 3.10, a and b) the system's combined stiffness will be equal to the sum of the stiffnesses of the individual elements, /90 and their deformation will be $x = \frac{P}{k_1+k_2}$. If the elastic elements are connected in series (Figure 3.10,c), the total deformation will be the sum of the deformations of each of the elements

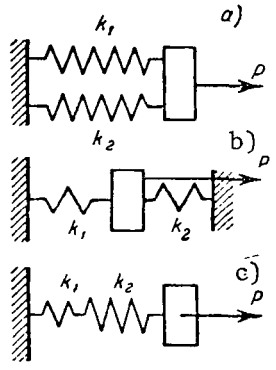

Figure 3.10.

$$x = x_1 + x_2 = P\left(\frac{1}{k_1} + \frac{1}{k_2}\right).$$

The reciprocal of an elastic element's stiffness is called its compliance. So, when several elastic elements are connected in series, it is not their stiffnesses but their compliances that combine.

All of the above characteristics and relations are the same for helical tension and compression springs. At the same time we should mention here some differences between those two types of element.

When an excessive load is placed upon a helical compression spring, its coils are pressed together, one against another. Naturally, the spring characteristic then loses its linearity. While the coils lying against one another often keeps the spring from breaking, it may also lead to vibroimpact states. Figure 3.11,a shows the elastic characteristic of such a spring.

Use of tension springs eliminates the possibility of impact interactions, but overloading may result in plastic strains and breakage. /91

Tension springs sometimes come pre-squeezed, as it is called. As part of the production process, their coils are squeezed tightly together, so that a finite stress has to be applied to their ends to overcome the squeeze. Figure 3.11,b shows the characteristic.

In designing mechanisms one naturally tries to satisfy those conditions that will assure that the elastic elements operate within the linear segments of their respective characteristics.

Some helical springs of special design have a nonlinear operating

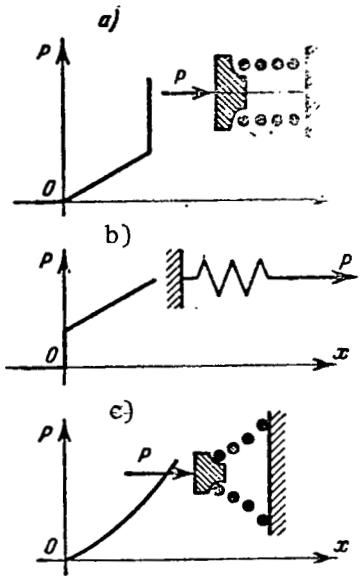

Figure 3.11.

characteristic. An example is a conical spring (Figure 3.11,c). We already know that the fewer a helical spring's coils and the smaller their diameter, the greater the spring's stiffness. When a load is applied to a conical spring, its bottom coils are pressed against the supporting surface, whereby the number of its working coils decreases. As the load is increased, more and more of the spring's coils are thus pressed out of operation, and its stiffness increases accordingly. The necessity for having elastic elements with these characteristics arises, for instance, in cases in which the frequency ω_n of the free oscillations of a system resting upon such springs has to remain roughly the same for different weights of the system.

As the load on a conical spring increases, its compression and stiffness increase accordingly. If for a linear characteristic we mean by stiffness $k = P/x$, then for a nonlinear spring with a characteristic $P = P(x)$ by stiffness we have to mean the derivative dP/dx calculated for a certain value of x.

In accordance with what we said in Chapter 1, the angular frequency of the oscillations of a load on a spring is expressed as

$$\omega_n = \sqrt{\frac{k}{M}}.$$

For a conical spring we consider approximately that

$$\omega_n = \sqrt{\frac{dP}{dx} \Big/ M}.$$

A conical spring can be so designed that its stiffness dP/dx will vary in proportion to the weight of the load upon it over its entire characteristic. The frequency of the system's free oscillations will then not be dependent on the magnitude of the load. Springs of this type are used in particular in certain vibration-insulating devices (see [87]). /92

In the mechanisms of monitoring and measuring instruments, and in those control systems, along with flat, spiral and helical springs, extensive use is made of the most varied elastic elements: manometer tubes and housings, aneroid cells, bellows, diaphragms, etc. We do not dwell here upon their calculation and design, which have been dealt with in a number of other studies (see, for example [65, 69, 77]).

§ 3.3. Vibrators

In machines and mechanisms operating on a vibratory or vibroimpact principle mechanical vibrators are widely used as oscillation exciters. Their basic element is a rotor bearing an unbalanced weight, made to rotate by an electric motor. The inertial forces of the unbalanced weight or weights are transmitted through the rotor bearings to those links of the mechanism or machine connected to the vibrator housing, and are the source of the excitation sustaining the vibratory or vibroimpact state.

The effect of the vibrator on the system being excited will vary, depending on the number of weights and on their relative positioning. With one unbalanced weight the exciting force, of constant magnitude, will rotate along with the weight. Two unbalanced weights make it possible to obtain either a variable force, constant in direction, or a variable moment.

In one of the vibration-testing machines (a vibration table put out by the German firm Losenhausen*) four vibrators were used. How that mechanism operates is shown schematically in Figure 3.12.

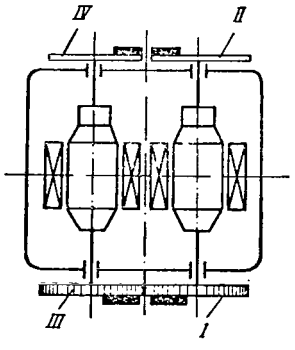

Figure 3.12.

Here two electric motors rotate four washers, to each of which a weight is fastened. The angular positioning of the weight can be varied by adjusting the machine. The electric motors turn in opposite directions, the shafts being additionally connected by a pair of gears which synchronizes the rotation of the four vibrators. This design makes it possible to obtain three different excitation patterns. When the vibrators are set as shown in Figure 3.13,a, the two weights on each of the rotors are fastened in phase with each other. The rotors are set with respect to one another so that all four weights (I, II, III, IV) are simultaneously in the up or down position. As a result, we get a variable force acting along the vibratory mechanism's z-axis. /93

When the setting is that of Figure 3.13,b, weights I and II are 180° out of phase, the same being true of III and IV. The rotors are set with respect to one another so that weights I and IV will be simultaneously in the up position while II and III are in the down position. The resultant effect of all four weights is a variable moment acting in the plane of the vibratory mechanism, i.e., with reference to the z-axis.

Finally, Figure 3.13,c shows the weights placed so as to generate a variable moment acting with respect to the x-axis. Here, as previously, weights I and II are 180° out of phase, also III and IV. But now the rotors are set so that I and III will be together in the up position, II and IV together in the down. Another vibratory mechanism also having four weights

* This German name, "recovered" from the Russian, is probably correct, but one cannot be entirely certain. Tr.

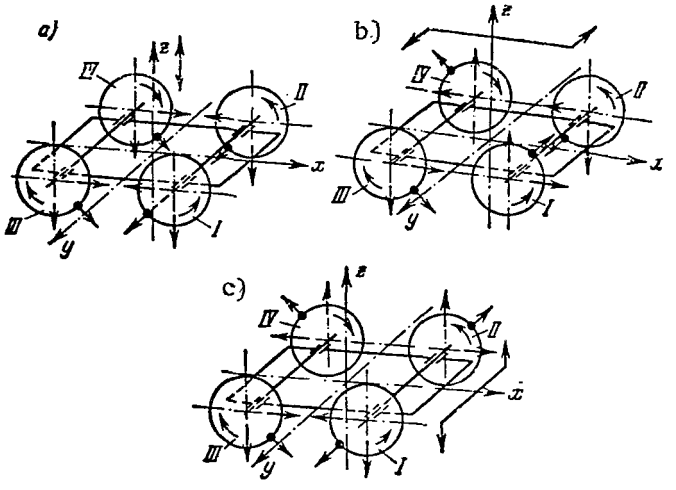

Figure 3.13.

has been described by V.A. Ivanovskiy [31].

We know of some vibration-testing machines utilizing vibratory mechanisms with four rotors and eight weights, driven by a single motor. A large number of vibrators is used in certain types of vibratory processing machinery, /94 with several electric motors and supplementary synchronizing devices in the form of chain drives or tooth gears, which mesh the movements of the individual rotors.

In the cases that we have considered all the weights revolve at the same rpm. There are some vibratory-mechanism designs in which the excitation pattern is polyharmonic, the rpm of the different weights being different. One such is described in the supplement.

Thus, by varying the rpm and positioning of the different weights, one can obtain virtually any excitation pattern.

In our analysis of the settings and operation diagrams of vibratory mechanisms, though we have spoken of the rpm of the vibrators, we have said nothing about how any one of them actually moves in the course of a single rotation. We tacitly assumed that each rotates absolutely evenly. Seldom is this really the case.

Let us attempt a more detailed analysis of the motion pattern of a /95 vibrator, for which we shall take as a model our old friend, the pendulum (Figure 3.14,a), whose length and mass are the dynamic equivalents of the unbalance and mass of a vibrator weight.

To avoid having to allow for the effect of gravity, we make the axis of the pendulum's rotation vertical. The free-oscillation frequency of such a pendulum is zero. For any value of the angle α it occupies a position of neutral equilibrium. We shall assume that the slider m_2 bearing the pendulum's rotation axis is fastened and remains stationary. If we now impart to the pendulum a certain angular velocity ω, it will rotate absolutely evenly (the friction in the suspension and the resistance of the medium being disregarded). The centrifugal force of inertia developed by the pendulum will be $P = m_1 L\omega^2$, its kinetic energy $T = \frac{m_1 L^2 \omega^2}{2}$.

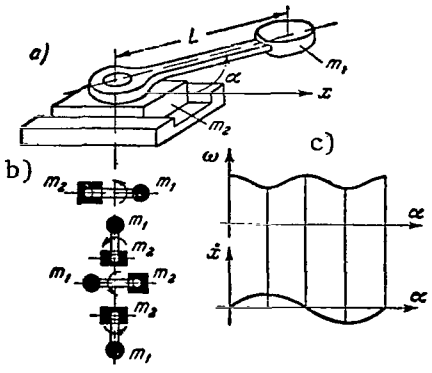

Figure 3.14.

Now we feel the slider, enabling it to move along the axis x - x. No external forces of any kind act upon the pendulum/slider system along that axis. We know that in such a case the system's center of inertia will not move along the axis x - x. So the slider m_2 will start to move. It will move back and forth in time with the pendulum's rotation, as shown in Figure 3.14,b. In freeing the slider, we did not alter the system's store of kinetic energy. But now this kinetic energy will be periodically redistributed in accordance with the equation for the system's energy balance, which we write as

/96

$$T = T_1 + T_2,$$
$$\frac{m_1 L^2 \omega^2}{2} = \frac{m_1 L^2 \omega^{*2}}{2} + m_2 \dot{x}^2, \qquad (3.21)$$

where T_1 is the kinetic energy of the rotating pendulum, T_2 the kinetic energy of the moving slider.

The slider's motion will not be even. At the outermost positions its velocity will be zero, reaching the maximum in the middle position. Then, in accordance with equation (3.21), the pendulum's angular velocity ω^* will also change, becoming equal to ω when the slider's velocity equals zero, and decreasing as the slider's velocity increases. Figure 3.14,c graphs the variations of the velocities ω^* and $\dot{x}$. It is not hard to understand that, the smaller the pendulum's mass compared to that of the slider, the smaller the latter's displacements and velocities will be, hence the smaller will be the fluctuations of the pendulum's angular velocity.

The motion of a vibrator presents roughly the same picture. If its housing remains stationary or undergoes relatively small displacements whose amplitude is small compared to the magnitude of the imbalance, the rotor may rotate quite evenly. But if the displacements of the vibrator housing become substantial, the fluctuations of the rotors' angular velocity will become appreciable, and in a number of cases these fluctuations will have to be allowed for (see, for example, [9, 11, 48]). Moreover, the effect of the oscillations of the vibrator housing on the motion of its rotor has now been thoroughly studied, and practical use is already being made of it in machines of the vibratory type [4, 13, 14].

Even so, in our further consideration of the dynamics and stability of the vibrational and vibroimpact states of motion of mechanisms with elastic couplings we shall always ascribe to a vibrator the property of directional effect, and we shall disregard the influence exerted upon it by the system being excited. For solving a number of important problems this assumption is quite admissible.

/97

Along with the mechanical vibrators that serve as oscillation exciters, electromagnetic, pneumatic and hydraulic vibrators and pulsators are used also. Their design is described and theory discussed in [26, 53, 59 and 68], among others.

We shall not dwell upon the dynamics of vibrators of different types, since we shall henceforward always assume that a vibrator's effect on the system being excited is known and is not affected by the motion of that system. We shall therefore be able to regard the motion of a mechanism with elastic couplings as though, applied to one or another of its links, were forces or moments of forces constituting known functions of the time, and most frequently we shall be dealing with cases wherein the vibratory mechanism generates a sinusoidal excitation.

§ 3.4. Damping

Allowing for the phenomenon of energy scattering or dissipation in the motion of a mechanism is very important. The reasons for the dissipation of energy, which winds up as heat, are: 1) friction forces within the kinematic pairs and external friction between the mechanism's links and the medium in which they are moving; 2) internal-friction forces inside the material of the elastic couplings, also friction forces arising at the places where the elements are in contact with immovable joints (an effect sometimes known as structural damping).

A large proportion in the over-all balance of the irretrievably dissipated energy may be that lost in impact interactions within kinematic pairs of a mechanism operating under conditions in which "breaks" occur in its kinematic linkage, followed by collisions between the latter's elements. Three chapters of this book are devoted to vibroimpact states of motion. In this /98
section, therefore, we shall discuss only those dissipative properties of a mechanism resulting from forces of external or internal friction.

The friction forces arising in the relative sliding of the elements of a kinematic pair usually belong to one of two classes of force: those of dry friction or those of fluid friction.

If a slider subjected to a force N is moving along an unlubricated slideway (Figure 3.15,a), to sustain its motion, we must apply to it a propulsive force F, which is to the force N as

$$F = f \cdot N. \tag{3.22}$$

The force F is called the force of dry friction, the coefficient f the coefficient of dry friction. The occurrence of this force is attributed to the interaction of the microroughnesses existing on the sliding surfaces. Observation reveals that, under conditions of interaction close to those of dry friction, the friction coefficient f does not much depend on the relative speed of slide of the surfaces. An exception is the range of extremely low

sliding speeds, close to the starting speed. At those speeds the friction coefficient increases, reaching its maximum at the starting speed (force of static friction) $f_0 > f$. If we neglect this feature, we can represent the characteristic of the dry-friction force as shown in Figure 3.15,b. As we see, it is a nonlinear function of the velocity of the relative motion of the elements of the kinematic pair. /99

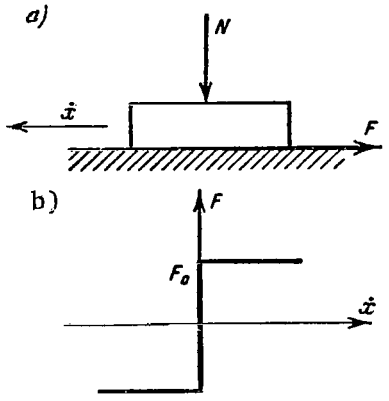

Figure 3.15.

Apart from anything else, the characteristic reflects the fact that with a velocity $\dot{x} = 0$ the force of friction can assume any value within the limits $-F_0 \leq F \leq F_0$.

In a case in which the rubbing surfaces are separated by a layer of lubricant, the friction takes on a fluid character. Since the sliding surfaces, supports, slideways and slide bearings of the overwhelming majority of machines and instruments are lubricated, we can see that allowing for the fluid friction acquires prime importance in solving the dynamic problems. Observation reveals that the forces of fluid friction are proportional to the relative speed of slide over a comparatively broad velocity range. This enables us, in allowing for the fluid friction, to use the linear characteristic shown in Figure 3.16 by considering that in these cases the force of friction is $F = -c\dot{x}$, where c is the coefficient of fluid friction.

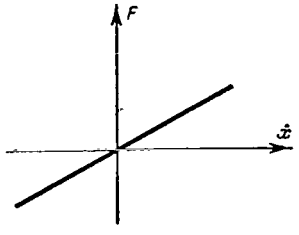

Figure 3.16.

A substantial portion of the over-all energy dissipated by a mechanism with elastic couplings in the process of its oscillation is due to the work of the internal-friction forces inside the elastic-coupling material, i.e., to the hysteresis losses, as they are called. These losses are attributed to the peculiarities of the diagram of the multiple loading and unloading of virtually any machine-building material. Figure 3.17a shows such a diagram. As we see there, for one and the same strain value the stress is somewhat greater when such multiple loadings and unloadings increase than when they decrease. This holds true even in a case in which the maximum stress does not exceed the proportionality limit. The closed curve thus obtained is called the hysteresis loop. The area bounded by that loop represents the amount of energy dissipated by a unit volume of material in the course of one cycle. When a spring is repeatedly subjected to tension and compression, owing to the hysteresis loop the pattern of the effect of the elastic force on the mass M (Figure 3.17b) will be such that, when the mass M moves to the right ($\dot{x} > 0$), the force /100 of the spring Q will be $(kx + R)$; when the mass M moves to the left ($\dot{x} < 0$), the elastic restoring force kx, there is always present in the system a force of resistance to motion, which produces a damping effect. The magnitude of this

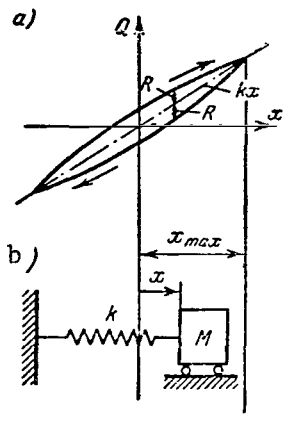

Figure 3.17.

force varies as a function of the system's displacement.

At the beginning of this section, as one of the factors contributing to energy dissipation we mentioned structural damping. As a rule, where mechanisms of machines and instruments are concerned, the relative weight of this factor is not great. We shall therefore not dwell upon it here, referring interested readers to the book that we list as [49].

In attempting to allow for this or that form of damping we must keep in mind that formal analysis of a system's forced oscillations is relatively simple only in the case of linear friction. Only then, as we shall see farther on, will the system's motion be strictly harmonic under the influence of a harmonic external force.

But in other cases, too, the damping of a system's motion actually differs little from the harmonic. Proceeding from this premise, we can easily estimate the effect of different kinds of damping by using a method based on finding the equivalent linear friction (see, for instance, § 8.7 below).

This method consists in determining, for this or that kind of damping, the power that the system dissipates during one oscillation cycle, then determining the coefficient of linear friction c_{equiv} corresponding to that power. The dissipation properties will henceforward be treated as in an ordinary linear system. /101

Another advantage of this method is that, by using it, we automatically allow for the correct sign of the friction force. Since any force of friction, wheather external or internal, is a resistance to motion, it must be directed against the velocity. A force possessing such a characteristic is the linear force of friction $c\dot{x}$. For any other kind of damping, the force's sign has to change when the velocity's sign changes, which is very convenient.

§ 3.5. Classifying Mechanisms with Elastic Couplings

As already mentioned, a mechanism with elastic couplings retains all of the properties of an ordinary mechanism. Keeping this in mind, we easily arrive at a generalized basic classification of mechanisms, one including both ordinary mechanisms and those with elastic couplings. We shall confine ourselves here to the bare essentials of such a classification, giving a few practical examples to illustrate its fundamental principles [110].

A classification of mechanisms with elastic couplings has to be based on the concept of a kinematic pair with an elastic coupling. If we confine ourselves to analysis of mechanisms with only lower pairs, we shall have only

to consider a sliding pair with an elastic coupling and a turning pair with an elastic coupling.

Figure 3.18 shows examples of three designs of a sliding pair with an elastic coupling: a) with the elastic element under tension, b) with the elastic element under compression, c) with the elastic element subjected to bending. Figure 3.18,d shows the characteristic of these kinematic pairs, plotted on the assumption that the elastic elements have not been preloaded. In all three cases, the elastic coupling acts in only one direction.

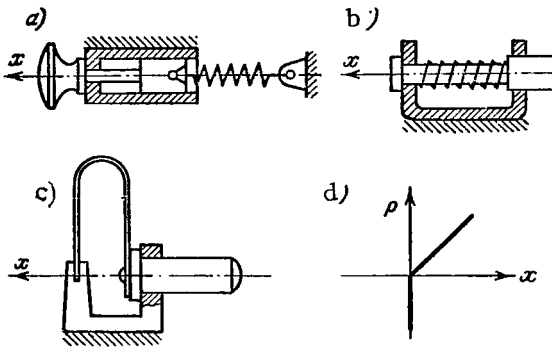

Figure 3.18.

By changing the design we could obtain kinematic pairs with elastic couplings acting in two directions and with different (linear and nonlinear) characteristics.

Figure 3.19 shows three designs of a turning pair with an elastic coupling: a) with a helical spring as the elastic element, b) with a spiral spring as the elastic element. In both cases the elastic coupling acts in only one direction. Figure 3.19, c shows a design of a turning pair with a preloaded elastic coupling that acts in two directions. The respective elastic characteristics of these kinematic pairs appear in Figure 3.19, d and e. /102 /103

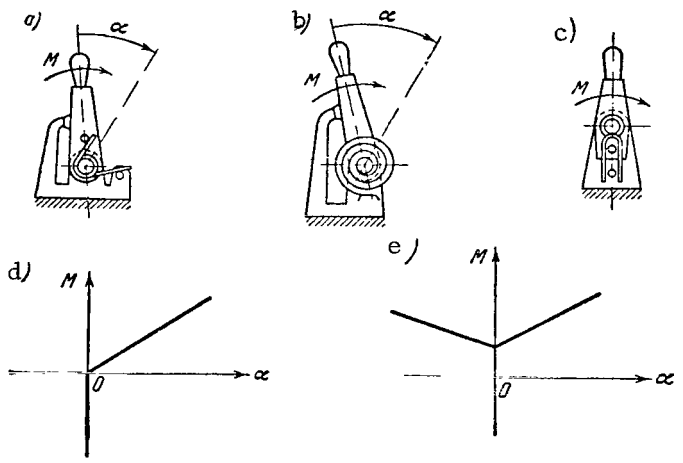

Figure 3.19.

In all the cases that we have examined, we have but a forced closure of the kinematic pair with the aid of the elastic couplings. The nature of the relative motion of its elements is determined entirely by their configuration. In addition to kinematic pairs of this type, widely used, too, are different designs of elastic joints in which the elastic couplings produce not only a forced interaction of the connected elements but also govern the nature of their relative motion.

Figure 3.20 shows three designs of such elastic joints: a) with a helical spring affording a translatory motion of the elements, b) with a helical spring affording a rotary motion of the elements, c) with a flat spring affording a rotary motion of the elements, and d) with a flat torsional spring affording a rotary motion.

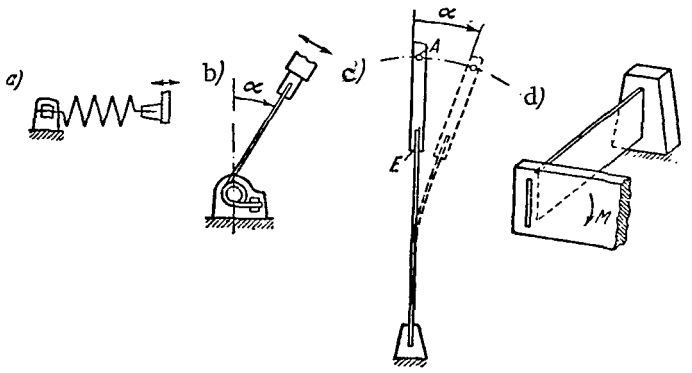

Figure 3.20.

Comparing kinematic pairs with an elastic coupling and elastic joints, we note the following.

The design of an elastic joint is simpler than that of a kinematic pair with an elastic coupling. The elastic joint permits relative mobility without gaps or clearances. At the same time the precision of the relative displacement of the elements of an elastic joint is much lower than that afforded by a kinematic pair. /104

A turning kinematic pair with an elastic coupling affords a high precision of relative motion of the elements with any kind of loading of them and for a large amplitude of that motion up to several complete turns. An elastic joint with, for example, a flat spring (Figure 3.20,c) assures a motion fairly close to rotational only for small deformations of the elastic element, the precision of that motion being largely dependent on the nature of the force effect [101].

These features of kinematic pairs with an elastic coupling and of elastic joints determine whether and how they will be used in the mechanisms of machines and instruments.

It must be said that the precision of the relative motion of the elements of an elastic joint can be greatly enhanced through use of a more complicated design. Thus, in particular, the so-called cross-spring pivot finds wide application. An example of its design, in which flat springs are used, is shown in Figure 3.21. Here the relative motion of the elements through small angles of rotation depends only on the magnitude of the displacement not on the nature of the load to which the elements are subjected.

As we see from the foregoing examples, kinematic pairs with elastic couplings and elastic joints can be classified with respect to: 1) type of relative motion of the joined links, 2) types of elastic element, and 3) characteristics of the elastic couplings. A classification of mechanisms with elastic couplings can be based on these same criteria. /105

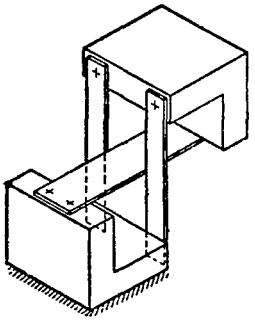

Figure 3.21.

Figure 3.22 shows three four-link mechanisms with elastic couplings. The hinged four-bar linkage (Figure 3.22,a) has an elastic coupling in the form of a helical spring joining the rocker to the support. Like an ordinary mechanism, it has two extreme positions, indicated in the figure by the dashed lines. We assume that the elastic coupling is to some extent prestressed. Then both the mechanism's extreme positions are positions of its equilibrium (we disregard the weight of the links, considering the mechanism to be situated in a horizontal plane). The position a' corresponds to the least deformation of the elastic coupling. If the crank is deflected from that position by a small angle and then let go, the mechanism will start to perform an oscillatory motion. So this position of the mechanism is one of stable equilibrium. Opposite to it is the position a" of unstable equilibrium. If removed from that position, the mechanism will not return to it but will go over to the position of stable equilibrium. Hence the elastic couplings impart to the mechanism the properties characteristic of any oscillating system, and it is these properties that primarily have to be allowed for in an analysis of the mechanism's dynamics and stability.

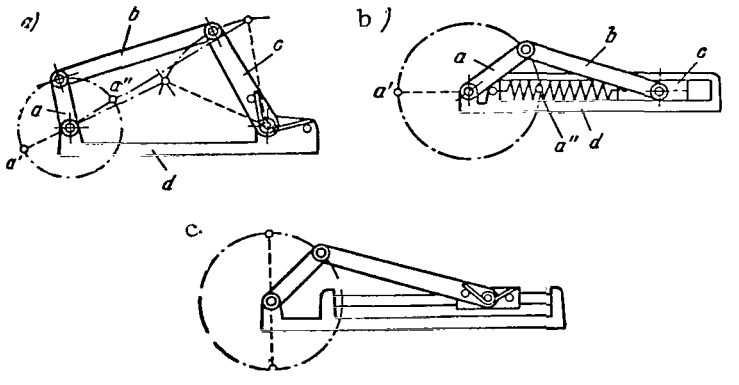

Figure 3.22.

/106

Figure 3.22,b shows a sliding-block linkage with a spring-actuated slider. Here the elastic coupling works under tension. It is easy to see that a' is thus the stable-equilibrium position, a" that of unstable equilibrium. In both these cases the elastic coupling has been superimposed on a kinematic pair connecting a moving link to a stationary one. Figure 3.22,c shows a case in which the elastic coupling connects two moving links.

We readily conceive that an exhaustive classification of mechanisms with elastic couplings can reflect the different classes of mechanism, the different modifications of them within each class, and the different numbers and types of elastic coupling and their respective characteristics. While such a classification would undoubtedly be of practical value, its compilation falls not within the ambit of the subject matter of this book.

To conclude this section, we cite several examples of the uses of mechanisms with elastic couplings in machines and instruments.

Figure 3.23,a shows a vibratory-conveyer mechanism. Essentially it is a variant of the mechanism in Figure 3.22,a, in which the kinematic pair with the elastic coupling has been replaced by an elastic joint comprising a series of flat springs.

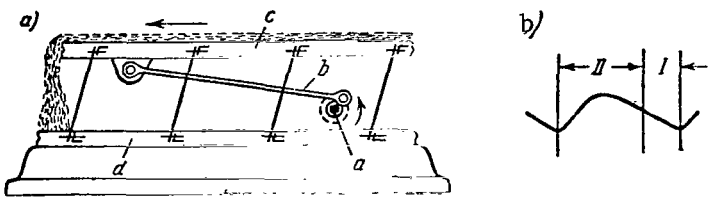

Figure 3.23.

Vibratory conveyers are used to move loose materials and piece items. As the crank turns, the trough moves back and forth. The flat springs are set at an angle of 10-15° to the vertical. As /107 a result, to a particle of the material m in the trough is imparted an acceleration directed at a certain angle to the horizontal, by virtue of which the particle's complete motion cycle breaks down into two stages (Figure 3.23,b). In stage I the particle is accelerated and moves along with the trough. At a certain instant it breaks loose from the trough and flies freely, in the direction in which the material is moving, until it falls back on the trough (stage II). A detailed theory of this process is set forth in [12].

In this design the motion of the trough is entirely determined by the parameters of the sliding-block linkage and by the rpm of the motor, in which case we have a so-called kinematic excitation. Along with this kinematic excitation, as we already know, extensive use is made of power excitation with the aid of special vibrators.

Figure 3.24 shows one such design of a vibratory trough. Here the vibrator mechanism is fastened to the frame m_1. Its two weights, by rotating, generate a harmonic force directed perpendicularly to the flat springs l_1 and l_2, which connect the frame m_1 with the trough m_2 and the trough with the base, respectively. The trough and frame are additionally connected by the helical springs f. This design assures that the base will be completely freed of vibration stresses when the masses m_1 and m_2 are so distributed that a line s_1 - s_2 connecting their centers of gravity coincides with the direction of their motion, and when the resultant of the forces developed by the springs f also coincides with that direction.

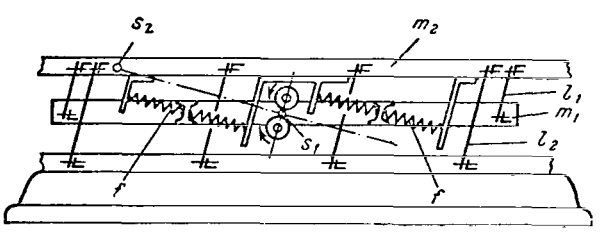

Figure 3.24.

The two examples that we have cited graphically illustrate the operating /108

principle of a very large category of machines designed for vibratory conveying, sorting, sifting, packing, testing, etc.

We shall now look at two other examples illustrating the use of mechanisms with elastic couplings in present-day instruments.

Figure 3.25 is a schematic of one of the many electromagnetic instruments used, in particular, for measuring voltages and currents. When the current is fed into the terminals 3, it passes through the winding of the coil 1, in which a magnetic field is created, and the plate 4 is drawn into the slit 2. Fastened to the instrument's spindle 5 is a pointer 6 bearing a counterweight 7. The elastic coupling 9, whose elastic element is a spiral spring, returns the pointer to its original position.

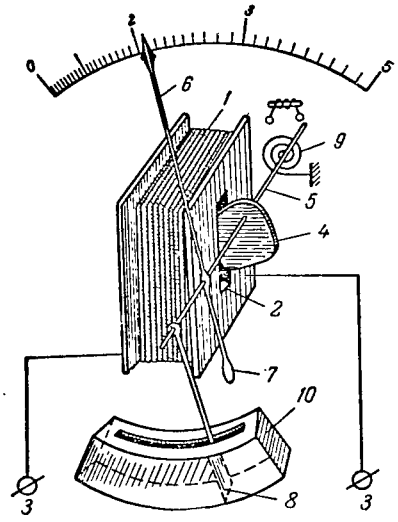

Figure 3.25.

In machines of the vibratory type one usually tries to reduce the internal friction to a minimum so as to cut down the energy consumption. In instruments, too, the friction must be minimal, and special measures are taken to achieve this.

In addition, for a large group of measuring instruments used for the most varied purposes, as well as for weighing and other devices, special supplementary dampers are required. We have selected just such an example. As we see, fastened to the instrument's spindle is the little piston 8 of an air damper 10. As it moves, the piston encounters resistance from the air, and this quickly damps the oscillations of the pointer. /109

Figure 3.26 is a schematic of an aeronautical instrument designed to measure the speed of motion of an airplane. This instrument contains two elastic elements: a pressure-gauge cell 1 and a spiral (hair) spring 5. The pressure-gauge cell is deformed as a function of the difference in the pressure outside it (P_2) and inside it (P_1). Its deformation moves a rigid center piece 6, which functions as the slider of a sliding-block linkage. Through the three-dimensional lever rotating about the axis x - x, the toothed sector 3, and the gear 4 this motion is transmitted to the instrument's pointer 7. The hair spring 5 forces the closing of the kinematic pair a consisting of the slider 6 and

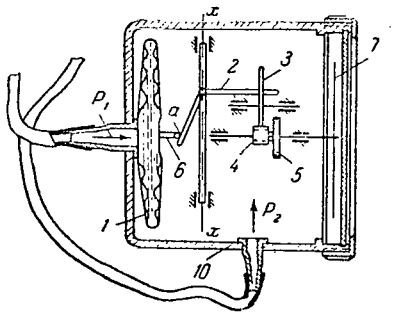

Figure. 3.26.

the lever 2.

Instruments like those that we have been introduced to are not intended to measure high-speed mechanical processes. In a number of cases, moreover, their dynamics and stability are of great interest, for in actual use they are frequently subject to various extraneous influences, which are then transmitted to the instrument's measuring system through the elastic couplings and are a source of sizable dynamic errors, to determine which we ultimately have to solve the equations of the motion of a mechanism with elastic couplings, then analyze the obtained solutions, just as we have to do in calculating machines of the vibratory type of instruments that are designed to measure high-speed processes.

CHAPTER 4

EQUATION FOR THE MOTION OF A MECHANISM WITH ELASTIC COUPLINGS

§ 4.1. The Mechanism's Static Characteristic. /110

If the external forces applied to the links of a mechanism are constant in magnitude and direction, when acted upon by these forces and by those of its elastic couplings, the mechanism will occupy a certain position, namely that of its static equilibrium (Figure 4.1). As the external forces evenly vary, the mechanism's static-equilibrium position will vary accordingly.

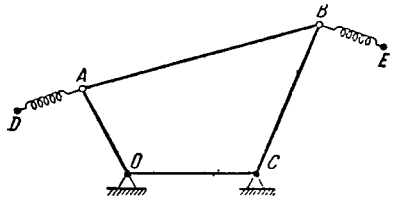

Figure 4.1.

One of the first problems that we face in designing a mechanism with elastic couplings is finding its equilibrium position as a function of the external forces acting upon it.

We shall determine the position of stable equilibrium for a case in which, of a number of external forces, a mechanism with elastic couplings is acted upon solely by the gravitational forces of its links (Figure 4.2).

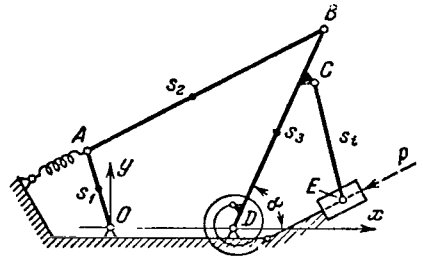

Figure 4.2.

We know that in a stable-equilibrium position the system's potential energy is minimal.

Taking as generalized coordinate the angle characterizing the position of any of the mechanism's links (e.g., α of the link BD), we can write the equilibrium condition as $\frac{d\Pi}{d\alpha} = 0$. /111 The mechanism's potential energy is comprised of the potential energy Π_I of the gravitational forces of the links and of the potential energy Π_{II} of deformation of the elastic couplings, so that its total value is $\Pi = \Pi_I + \Pi_{II}$. Allowing for the gravitational forces G_i applied at the centers of gravity s_i of the links, and designating the ordinates of the centers of gravity as y_i, we can write the component Π_I as:

$$\Pi_I = \sum_{i=0}^{n} G_i y_i + \Pi_0, \tag{4.1}$$

where Π_0 is the initial store of potential energy, whose magnitude depends on the selected point of origin. The values of the ordinates y_i depends only on the dimensions and position of the mechanism, i.e., on the value of the generalized coordinate, and can be written as $y_i = y_i(\alpha)$.

To determine the mechanism's second potential-energy component, we have to know the characteristics of each of the m elastic couplings acting upon its links. In the general case these characteristics are nonlinear and are expressed by the relation $F_i = F(\delta_i)$, where δ_i is the magnitude of the deformation of the i-th elastic coupling.

Defining the potential energy of deformation as the work done by the elastic forces in returning the couplings to the undeformed state, we get /112

$$\Pi_{\Pi} = \sum_{i=0}^{m} \int_{\delta_i}^{0} F(\delta_i)\, d\delta_i. \tag{4.2}$$

The amounts of deformation of the elastic couplings for given link dimensions depend solely on the mechanism's position $\delta_i = \delta_i(\alpha)$, whence

$$d\delta_i = \frac{d\delta_i}{d\alpha} \cdot d\alpha. \tag{4.3}$$

Substituting (4.3) in (4.2) and changing the integration limits accordingly, we get

$$\Pi_{\Pi} = \sum_{i=0}^{m} \int_{\alpha}^{\bar{\alpha}} F(\delta_i) \frac{d\delta_i}{d\alpha} d\alpha, \tag{4.4}$$

where $\bar{\alpha}_i$ is the value of the generalized coordinate for the mechanism position in which the coupling i is not deformed.

Combining (4.1) and (4.4), we get the total potential energy of a mechanism with elastic couplings being acted upon by the gravitational forces of its links:

$$\Pi = \sum_{i=0}^{n} G_i y_i + \sum_{i=0}^{m} \int_{\alpha}^{\bar{\alpha}} F(\delta_i) \frac{d\delta_i}{d\alpha} d\alpha. \tag{4.5}$$

Differentiating (4.5) with respect to the generalized coordinate, we get

$$\frac{d\Pi}{d\alpha} = \sum_{i=0}^{n} G_i \frac{dy_i}{d\alpha} + \sum_{i=0}^{m} F(\delta_i) \frac{d\delta_i}{d\alpha} = -M_{\Pi},$$

where M_{Π} is the mechanism's generalized moment.

The condition for the mechanism's equilibrium is written as:

$$\sum_{i=0}^{n} G_i \frac{dy_i}{d\alpha} + \sum_{i=0}^{m} F(\delta_i) \frac{d\delta_i}{d\alpha} = 0.$$

The roots of this equation give the values of the generalized coordinate for the equilibrium position of a mechanism being acted upon by gravitational forces.

Apart from gravitational forces, the links of a mechanism may be subjected to other forces which, while not potential in nature, yet do either remain constant in magnitude and direction or vary relatively slowly during the time in question*. To determine a mechanism's equilibrium position as influenced by these forces, we use the general equilibrium condition, according to which a mechanism's generalized moment is zero in the equilibrium position. /113

To simplify the calculation, we shall assume that acting upon our mechanism is only the one force P, constant in direction and constant or "slowly" varying in magnitude (see Figure 4.2). The generalized moment of this force is

$$M_P = P \frac{ds_P}{d\alpha},$$

where S_P is the slider displacement.

With allowance made for the effect of the potential forces, the total generalized moment is:

$$M = P \frac{ds_P}{d\alpha} - \sum_{0}^{n} G_i \frac{dy_i}{d\alpha} - \sum_{0}^{m} F(\delta_i) \frac{d\delta_i}{d\alpha}. \tag{4.6}$$

So for determining the mechanism's equilibrium position we have at our disposal the equation

$$P \frac{ds_P}{d\alpha} - \sum_{0}^{n} G_i \frac{dy_i}{d\alpha} - \sum_{0}^{m} F(\delta_i) \frac{d\delta_i}{d\alpha} = 0. \tag{4.7}$$

The value of α determined from this equation and the mechanism's corresponding stable-equilibrium position we designate as α_0.

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* In which case the forces acting on a mechanism's links can be regarded as "slowly" varying is a question that we shall take up later.

The relation $\alpha_0 = \alpha_0(P)$ linking the mechanism's successive equilibrium positions with the magnitudes of the external constant or "slowly" varying forces acting upon its moving links, we call the mechanism's static characteristic. /114

We should mention that equation (4.7) may have roots for both the stable and unstable equilibrium positions. Even so, and notwithstanding the equation's relative complexity, approximate determination of the root is usually simplified by the fact that, with a good choice of the generalized coordinate, the range of the latter's variation is not very great, the limits being established in the process of designing the mechanism.

If an approximate value of the root α is determined by any means (e.g., graphic), it is desirable that further refinement of the equation be by the method of linear corrections. Then the value of the correction $\Delta\alpha$ can be found (Figure 4.3) from the obvious relation

$$\Delta\alpha = M' \Big/ \left(\frac{dM}{d\alpha}\right)_{\alpha=\alpha'}.$$

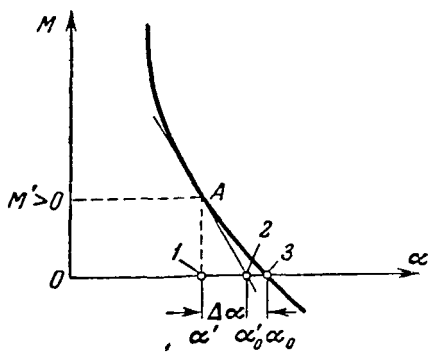

Figure 4.3.

If further refinement is necessary, these calculations must be repeated, with substitution in (4.7) of the value

$$\alpha = \alpha'_0 = \alpha' + \Delta\alpha$$

The advisability of using the linear-correction method for seeking α_0 stems from the fact that the analytical expression $dM/d\alpha$, needed for determining $\Delta\alpha$, is used in the subsequent calculations also.

§ 4.2. The Mechanism's Free Oscillations.

We move a mechanism out of the stable-equilibrium position by imparting to some point of any of its links an initial deviation or initial velocity. If the initial perturbation is sufficiently small, its effect will be that the mechanism will perform about the equilibrium position a periodic motion with a certain frequency. This motion can be determined by imparting to the generalized coordinate α_0 a small increment ε and setting up in Lagrangian form an equation for the mechanism's small oscillations, in the process of which $\alpha = \alpha_0 + \varepsilon$, where always $\varepsilon \leq \varepsilon_0$: /115

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varepsilon}}\right) - \frac{\partial T}{\partial \varepsilon} = M(\varepsilon). \qquad (4.8)$$

As in determining the static-equilibrium position, we shall be assuming that with the exception of the forces of gravity only one external force P, whose value remains constant throughout the interval of interest, acts on the mechanism.

In the process of small oscillations of the mechanism, its kinetic and potential energies, as well as the generalized power moment P vary in the function of the small parameter ε.

The expression of the generalized moment M in the function of the generalized coordinate α was obtained above in the determination of the position of equilibrium of the mechanism

$$M(\alpha) = P\frac{ds_P}{d\alpha} - \frac{d\Pi}{d\alpha}.$$

Substituting value $\alpha = \alpha_0 + \varepsilon$ in this equation, we get

$$M = M(\alpha_0 + \varepsilon).$$

Bearing in mind that we are examining small oscillations of the mechanism, and that consequently ε is a small parameter, we expand this function into Taylor's series with respects to the powers of ε:

$$M(\alpha_0+\varepsilon) = M_0 + \left(\frac{dM}{d\alpha}\right)_0 \varepsilon + \left(\frac{d^2M}{d\alpha^2}\right)_0 \frac{\varepsilon^2}{2} + \dots$$

The "zero" indices at the constant term and the derivatives indicate that the magnitudes of these values should be determined for the static equilibrium position of the mechanism.

As the result of the expansion of the two components of the generalized moment, we get /116

$$P\frac{ds_P}{d\alpha} = P\left[\left(\frac{ds_P}{d\alpha}\right)_0 + \left(\frac{d^2s_P}{d\alpha^2}\right)_0 \varepsilon + \left(\frac{d^3s_P}{d\alpha^3}\right)_0 \frac{\varepsilon^2}{2} + \dots\right],$$
$$\frac{d\Pi}{d\alpha} = \left(\frac{d\Pi}{d\alpha}\right)_0 + \left(\frac{d^2\Pi}{d\alpha^2}\right)_0 \varepsilon + \left(\frac{d^3\Pi}{d\alpha^3}\right)_0 \frac{\varepsilon^2}{2} + \dots$$

According to condition (4.7) in the equilibrium position the following equation should exist:

$$P\left(\frac{ds_P}{d\alpha}\right)_0 - \left(\frac{d\Pi}{d\alpha}\right)_0 = 0.$$

Disregarding, owing to the small value of ε, the terms containing its

square, we obtain the right-hand member of the Lagrangian equation in the following form:

$$M \approx \left(\frac{dM}{d\alpha}\right)_0 \varepsilon = -\left[\left(\frac{d^2\Pi}{d\alpha^2}\right)_0 - P\left(\frac{d^2 s_P}{d\alpha^2}\right)_0\right]\varepsilon. \tag{4.9}$$

We designate the difference with

$$\left[\left(\frac{d^2\Pi}{d\alpha^2}\right)_0 - P\left(\frac{d^2 s_P}{d\alpha^2}\right)_0\right] = K. \tag{4.10}$$

In the future, value K will be called the quasielastic coefficient of the mechanism. From (4.10) it follows that the value of the quasielastic coefficient depends on the characteristics of the elastic couplings, on the values of the forces of gravity of the components, on the configuration and position of the mechanism.

We shall now set up the left-hand member of the Lagrangian equation, for which purpose we have to find the expression of kinetic energy T of the mechanism as a function of the small parameter ε. As we know, in the general case, the kinetic energy of the flat mechanism is determined by the following relationship:

$$T = \frac{\omega^2}{2}\sum^{n}\left[m_i\left(\frac{ds_i}{d\alpha}\right)^2 + J_i\left(\frac{d\alpha_i}{d\alpha}\right)^2\right], \quad \omega = \frac{d\alpha}{dt}, \tag{4.11}$$

where $\frac{d\alpha_i}{d\alpha}$ and $\frac{ds_i}{d\alpha}$ are functions of α, characterizing the transmitting relationships of the mechanism. Expanding these functions into Taylor's series with respect to ε, we get /117

$$\left.\begin{aligned}\frac{ds_i}{d\alpha} &= \left(\frac{ds_i}{d\alpha}\right)_0 + \left(\frac{d^2 s_i}{d\alpha^2}\right)_0 \varepsilon + \left(\frac{d^3 s_i}{d\alpha^3}\right)_0 \frac{\varepsilon^2}{2} + \dots,\\ \frac{d\alpha_i}{d\alpha} &= \left(\frac{d\alpha_i}{d\alpha}\right)_0 + \left(\frac{d^2\alpha_i}{d\alpha^2}\right)_0 \varepsilon + \left(\frac{d^3\alpha_i}{d\alpha^3}\right)_0 \frac{\varepsilon^2}{2} + \dots\end{aligned}\right\} \tag{4.12}$$

Bearing in mind that

$$\omega = \frac{d(\alpha_0 + \varepsilon)}{dt} = \dot{\varepsilon},$$

and limiting ourselves to accounting for only the first terms of expansions (4.12), we can present the kinetic energy expression in the following form:

$$T = \frac{\dot{\varepsilon}^2}{2}\sum_{0}^{n}\left[m_i\left(\frac{ds_i}{d\alpha}\right)_0^2 + J_i\left(\frac{d\alpha_i}{d\alpha}\right)_0^2\right]. \tag{4.13}$$

We designate the sum with

$$\sum_0^n \left[m_i \left(\frac{ds_i}{d\alpha} \right)_0^2 + J_i \left(\frac{d\alpha_i}{d\alpha} \right)_0^2 \right] = J. \tag{4.14}$$

In the future we shall term value J the inertial coefficient* of the mechanism. The value of the inertial coefficient, as well as the value of the quasielastic coefficient depends on the magnitudes of the masses, moments of inertia, and dimensions of the components, and on the position of the mechanism.

Differentiating expression (4.13) with respect to ε and $\dot{\varepsilon}$, and then with respect to t, we shall get terms contained in the left member of the Lagrangian equation. It is not hard to see that in this case

$$\frac{\partial T}{\partial \varepsilon} = 0, \quad \frac{\partial T}{\partial \dot{\varepsilon}} = J\dot{\varepsilon}$$

and finally

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varepsilon}} \right) = J\ddot{\varepsilon}. \tag{4.15}$$

Substituting (4.9) and (4.15) in (4.8) we get the equation of the motion /118 of the mechanism near the position of equilibrium

$$J\ddot{\varepsilon} + K\varepsilon = 0. \tag{4.16}$$

The solution of this equation, by analogy with (2.9) has the following appearance

$$\varepsilon = C_1 \cos \omega_n t + C_2 \sin \omega_n t,$$

where ω_n is the angular frequency of the free oscillations of the mechanism.

Since the initial perturbation consisted of the fact that increment ε_0 was given to the generalized coordinate, using the initial conditions of motion $\varepsilon = \varepsilon_0$, $\dot{\varepsilon} = 0$ when $t = 0$, we get

$$\varepsilon = \varepsilon_0 \cos \omega_n t.$$

Consequently, the mechanism, taken out of the position of static

---

* The inertial coefficient in the case of small oscillations of the mechanism represents a constant value in distinction from the moment of inertia given (see equation (4.11)) where the derivatives have no index characterizing the static equilibrium position.

equilibrium, performs harmonic oscillations with respect to this position. Their frequency depends on the ratio between the quasielastic and inertial coefficients of the mechanism and is determined by the usual relationship

$$f_n = \frac{1}{2\pi}\sqrt{\frac{K}{J}} = \frac{\omega_n}{2\pi}. \qquad (4.17)$$

From equation (4.14) it follows that the inertial coefficient under all conditions is a substantially positive value, from equation (4.17) it follows that in order that the mechanism, having been taken out of the position of equilibrium, would be able to perform an oscillating motion, the quasielastic coefficient near the position of equilibrium must also be a substantially positive value. Only under this condition would the position of the static equilibrium of the mechanism be stable. In the opposite case the oscillation frequency would be either an imaginary value (if $K < 0$), or be equal to zero (if $K = 0$), which would correspond to the positions of unstable or indifferent equilibrium of the mechanism.

Earlier, we have established that the values of coefficients K and J depend on the generalized coordinate α_0 characterizing the position of the static equilibrium of the mechanism*.

The position of the static equilibrium of a mechanism may vary in accordance with the system of external constants or "slowly" changing forces, applied to its components. Since K and J vary in this case to a different extent, then consequently, the frequency of the free oscillations of a mechanism varies within the function of the generalized coordinate. /119

Relationship $f_n = f_n(\alpha_0)$ or $\omega_n = \omega_n(\alpha_0)$, connecting the frequency of the free oscillations of the mechanism with the position of static equilibrium (Figure 4.4) will be called the characteristic of the frequency of free oscillations of the mechanism. Knowing this characteristic we can solve the problem about which of the external forces, applied to the mechanism, and which time functions, may be considered "slowly" changing.

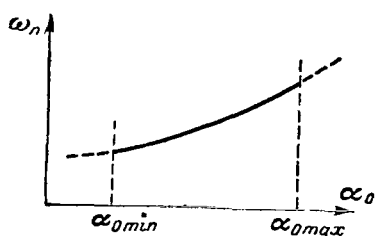

Figure 4.4.

If we designate the frequency of a certain periodic force applied to a mechanism, with ω, then we may disregard the dynamic effect of this force, considering it slowly-changing only under the condition that $\omega_n >> \omega$, i.e., under the condition that the mechanism's own oscillation frequency is considerably greater than the frequency of the external force.

* And do not depend on ε, i.e., remain constant within the bounds of small oscillations of the mechanism.

Thus, the dynamic effect of any one periodic force depends first of all on the properties of the mechanism to which this force is applied.

§ 4.3. Motion of a Mechanism Under the Action of a Pulsing Force.

Before turning to the direct setting-up of the equation of motion of a mechanism under the action of a pulsing force, we make two remarks, pertaining to this and the next paragraphs:

1) Earlier it was shown that the amplitude of the free motions of a mechanism is determined by the value of the initial perturbation, and the entire analysis was carried out in the assumption of the smallness of this perturbation, and consequently, of the smallness of the amplitude of free oscillations.

Similarly, in the investigation of forced oscillations of the mechanism /120
we shall assume that the amplitude of the periodic actuation, acting on the mechanism (amplitude of pulsation or vibration), remains small. The condition of the smallness of the actuation amplitude, while it is indispensable, at the same time might prove an insufficient condition for the smallness of the amplitude of the forced oscillations of the mechanism. Nevertheless, we shall set up the equation of the mechanism's motion, preceeding from the assumption of the smallness of the amplitude, and in the future will establish the conditions under which this assumption remains in force.

2) In the preceding paragraph it was shown that free oscillations of a mechanism are performed near the position of its static equilibrium according to the harmonic law, so that

$$\frac{1}{T}\int_0^T \alpha\, dt = \alpha_0,$$

where

$$\alpha = \alpha_0 + \varepsilon.$$

In the investigation of forced oscillations we present the generalized coordinate in the form of the sum

$$\alpha = \alpha_{dyn} + \varepsilon. \tag{4.18}$$

signifying by the term α_{dyn} a certain "mean" value of the generalized coordinate, established in the process of forced oscillations of the mechanism, i.e.,

$$\alpha_{dyn} = \frac{1}{T}\int_0^T \alpha\, dt.$$

In case the forced oscillations, remaining small, prove aperiodic, this "mean" value of the generalized coordinate may be found from the equation

$$\alpha_{dyn} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \alpha\, dt. \tag{4.19}$$

The position, characterized by the value α_{dyn}, in the future will be called the *position of dynamic equilibrium* of the mechanism.

From equations (4.18) and (4.19) it follows that /121

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \varepsilon\, dt = 0.$$

In this way, in investigating the forced oscillations of a mechanism we shall assume that in the general instance they occur near a certain position of the dynamic equilibrium of the mechanism, which might not coincide with the previously-determined position of static equilibrium.

The relationships between the parameters of the mechanism, the parameters of actuation, and the position of the dynamic equilibrium of the mechanism will be established in the next chapter, while here we shall set up the equation of motion of a mechanism with elastic couplings, which is under the effect of a pulsing force.

Let the mechanism be acted upon, in addition to the force of gravity of its components, by only one external force P_t, differing from the constant or smoothly varying force P by the presence of the periodic component

$$P_t = P + \Phi(\omega t),$$

this periodic component being such that

$$\frac{1}{T} \int_0^T \Phi(\omega t)\, dt = 0,$$

where T is the period of function $\Phi(\omega t)$.

In other words, let us assume that the external force P_t pulses with a specific frequency ω near a constant or "slowly"-changing mean value. Let the mechanism, under the effect of this force perform, near a certain position of dynamic equilibrium, small oscillations, the equation of which we shall set up in the Lagrangian form, as before.

The generalized moment of the mechanism, by analogy with equation (4.6)

may be written as follows:

$$M = M(\alpha_{dyn} + \varepsilon) = [P + \Phi(\omega t)] \frac{ds_P}{d\alpha} - \frac{d\Pi}{d\alpha}.$$

Having expanded functions $\frac{ds_P}{d\alpha}$ and $\frac{d\Pi}{d\alpha}$ into Taylor's series with respect to the small parameter, and limiting ourselves to the first two terms of the expansion, we get /122

$$M \approx P\left(\frac{ds_P}{d\alpha}\right)_{dyn} - \left(\frac{d\Pi}{d\alpha}\right)_{dyn} + \left(\frac{ds_P}{d\alpha}\right)_{dyn}\Phi(\omega t) + \\ + \left\{[P + \Phi(\omega t)]\left(\frac{d^2 s_P}{d\alpha^2}\right)_{dyn} - \left(\frac{d^2\Pi}{d\alpha^2}\right)_{dyn}\right\}\varepsilon.$$

The indices with the constant term and derivatives, indicate that the values of these quantities should be determined for the so-far unknown position of the dynamic equilibrium of the mechanism, while, as we can easily see, difference

$$P\left(\frac{ds_P}{d\alpha}\right)_{dyn} - \left(\frac{d\Pi}{d\alpha}\right)_{dyn} = D$$

represents the component of the generalized moment, arising as the result of the difference between the positions of the static and dynamic equilibrium.

In the brackets, we separate the difference

$$P\left(\frac{d^2 s_P}{d\alpha^2}\right)_{dyn} - \left(\frac{d^2\Pi}{d\alpha^2}\right)_{dyn} = K_{dyn}.$$

Now the right member of equation (4.8) is equal to

$$M(\varepsilon) = D + \left(\frac{ds_P}{d\alpha}\right)_{dyn}\Phi(\omega t) - \left[K_{dyn} - \left(\frac{d^2 s_P}{d\alpha^2}\right)_{dyn}\Phi(\omega t)\right]\varepsilon. \qquad (4.20)$$

As for the left member of equation (4.8) we can, limiting ourselves to accounting for the terms containing $\dot{\varepsilon}^2$, write it according to equation (4.15) in the following way

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varepsilon}}\right) - \frac{\partial T}{\partial \varepsilon} = J_{dyn}\ddot{\varepsilon}, \qquad (4.21)$$

where

$$J_{dyn} = \sum_0^n \left[m_i\left(\frac{ds_i}{d\alpha}\right)^2_{dyn} + J_i\left(\frac{d\alpha_i}{d\alpha}\right)^2_{dyn}\right].$$

Having substituted (4.20) and (4.21) in (4.8) and divided the left and right members by J_{dyn}, we obtain the equation of motion of the mechanism subjected to a pulsating force, in the following form /123

$$\ddot{\varepsilon}+\left[\omega_{n\,dyn}^2-\frac{1}{J_{dyn}}\left(\frac{d^2s_P}{d\alpha^2}\right)_{dyn}\Phi(\omega t)\right]\varepsilon=d+\frac{1}{J_{dyn}}\left(\frac{ds_P}{d\alpha}\right)_{dyn}\Phi(\omega t), \tag{4.22}$$

where $\omega_{n\ dyn}=\sqrt{\frac{K_{dyn}}{J_{dyn}}}$ is the frequency of free oscillations of the mechanism near the position of dynamic equilibrium, and

$$d+\frac{D}{J_{dyn}}.$$

Here, we should note that the equation of motion of the mechanism would have an analogous form in the case of action on its components not of one but of several forces pulsating with a different frequency. In this case, evidently, instead of terms

$$\left(\frac{d^2s_P}{d\alpha^2}\right)_{dyn}\Phi(\omega t) \text{ and } \left(\frac{ds_P}{d\alpha}\right)_{dyn}\Phi(\omega t),$$

we should have substituted in equation (4.22) the sums

$$\sum_{i=1}^{k}\left(\frac{d^2s_{P_i}}{d\alpha^2}\right)_{dyn}\Phi_i(\omega t) \text{ and } \sum_{i=1}^{k}\left(\frac{ds_{P_i}}{d\alpha}\right)_{dyn}\Phi_i(\omega t),$$

which take into account the actions of all these forces.

If the pulsation of the force has a harmonic character, equation (4.22) attains the following appearance:

$$\ddot{\varepsilon}+\left[\omega_{n\,dyn}^2-\frac{P_0}{J_{dyn}}\left(\frac{d^2s_P}{d\alpha^2}\right)_{dyn}\cos\omega t\right]\varepsilon=d+\frac{P_0}{J_{dyn}}\left(\frac{ds_P}{d\alpha}\right)_{dyn}\cos\omega t. \tag{4.23}$$

Equations (4.22) and (4.23) represent inhomogeneous linear differential equations of the second order with periodic coefficients.

A homogeneous equation corresponding to equation (4.22) is a Hill's equation; a homogeneous equation corresponding to equation (4.32) is a Mathieu equation.

In this way, in the general case, when the transmission ratios of a mechanism are functions of its position, although the small forced oscillations of the mechanism do satisfy the linear differential equations, the coefficients /124

in these equations are found to be inconstant values and are determined by the character of the actuation. In particular cases only (for example for the mechanism consisting of round cog wheels) the value $(d^2s_p/d\alpha^2)_{dyn}$ may prove equal to zero. In this case (4.22) and (4.23) turn into equations with constant coefficients, the solution of which is well known.

Thus, for example, the full solution of the equation with constant coefficients, obtained from (4.23) when $(d^2s_p/d\alpha^2)_{dyn} = 0$, consists of the general solution of the homogeneous equation and a particular solution of the inhomogeneous equation. Dropping in the general solution the term corresponding to the free oscillations of the mechanism, we can write the equation of motion in the following form:

$$\varepsilon = \frac{P_0\left(\frac{ds_P}{d\alpha}\right)_0}{J(\omega_n^2 - \omega^2)}\cos\omega t.$$

Consequently, in this particular case, under the action of the pulsating force, the mechanism performs harmonic oscillations, the amplitude of which consists of the pulsation amplitude, the mechanism's parameters, and the ratio between the actuation frequency and the free oscillation frequency, these oscillations taking place near the static equilibrium position.

If the movement of the mechanism is described by a differential equation with constant coefficients, the positions of the static and dynamic equilibrium coincide.

If in this case $\omega_n >> \omega$, i.e., if to the mechanism we apply a "slowly"-developing force, the motion of the mechanism satisfies with a sufficient degree of accuracy the following equation

$$\varepsilon \approx \frac{P_0}{K}\left(\frac{ds_P}{d\alpha}\right)_0'\cos\omega t.$$

Example. In Figure 4.5 we show a four-component mechanism with an elastic bond, the torsion rigidity of which is constant and equal to q. Considering the masses of the mobile components small in comparison with mass M, concentrated in point A, we set up the equation of the small oscillations of the mechanism under the action of the pulsating force $\Phi(\omega t)$, applied to the slider. /125

First, we determine the static equilibrium position (for $\Phi(\omega t) = 0$), for which we set up the equation of the generalized moment of the mechanism. We select angle α as the generalized coordinate.

The potential energy of the mechanism examined is composed of the potential energy of mass M and the potential energy of deformation of the elastic coupling:

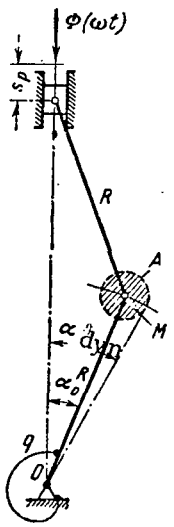

Figure 4.5.

$$\Pi = \Pi_0 + GR\cos\alpha + \frac{q\alpha^2}{2},$$

from which

$$\frac{d\Pi}{d\alpha} = -GR\sin\alpha + q\alpha.$$

Consequently the static-equilibrium of the mechanism is characterized by the root of the transcendent equation

$$\alpha_0 - \frac{GR}{q}\sin\alpha_0 = 0.$$

Let us determine the frequency of the mechanism's free oscillations, for which we shall calculate the quasi-elastic and inertial coefficients. According to the condition adopted in the example, the constant component of the external force P ≡ 0. Therefore

$$P\left(\frac{d^2s_P}{d\alpha^2}\right)_0 \equiv 0, \qquad \left(\frac{d^2\Pi}{d\alpha^2}\right)_0 = q - GR\cos\alpha_0.$$

Consequently, the quasielastic coefficient, according to equation (4.10) is equal to

$$K = q - GR\cos\alpha_0.$$

The inertial coefficient represents a constant value, equal to /126

$$J = MR^2,$$

As the result, the frequency of the free oscillations of the mechanism proves equal to

$$\omega_n = \sqrt{\frac{q - GR\cos\alpha_0}{MR^2}}.$$

Its value depends on the parameters of the mechanism and the equilibrium position which it occupies. Being deflected from this position by a small value, the mechanism performs harmonic oscillations near it, under the condition that

$$q > GR\cos\alpha_0.$$

Now we apply to the slider a pulsing force $\Phi(\omega t)$ and set up, according to (4.22) the equation of forced oscillations of the mechanism near the position of the dynamic equilibrium of the mechanism.

For this we must determine values D, $\left(\frac{ds_\Phi}{d\alpha}\right)_{dyn}$, and $\left(d^2s_\Phi/d\alpha^2\right)_{dyn}$ it follows that

$$s_\Phi = 2R\ (1 - \cos\ \alpha_{dyn});$$

in this case

$$\left(\frac{ds_\Phi}{d\alpha}\right)_{dyn} = 2R\sin\alpha_{dyn},$$
$$\left(\frac{d^2s_\Phi}{d\alpha^2}\right)_{dyn} = 2R\cos\alpha_{dyn}.$$

The constant component D for the generalized moment in the example examined by us is about equal to

$$D = GR\ \sin\ \alpha_{dyn} - q\alpha_{dyn}.$$

Consequently, the equation of the forced oscillations of the mechanism portrayed in Figure 4.5, has the following appearance:

$$\ddot{\varepsilon} + \left[\frac{q - GR\cos\alpha_{dyn}}{MR^2} - \frac{2\cos\alpha_{dyn}}{MR}\Phi(\omega t)\right]\varepsilon = -\frac{GR\sin\alpha_{dyn} - q\alpha_{dyn}}{MR^2} - \frac{2\sin\alpha_{dyn}}{MR}\Phi(\omega t).$$

If the position of the static equilibrium of the mechanism coincides with the vertical position, then $\alpha_0 = 0$ (when $q \neq 0$). Then, on the basis of the concepts proceeding from the symmetry of the mechanism, we may assume that the forced oscillations occur near this position. In other words, we may assume, that the positions of the static and dynamic equilibrium of the mechanism, which is in the extreme position. coincide, i.e., /127

$$\alpha_{dyn} = \alpha_0 = 0.$$

Here, the equation of motion of the mechanism assumes the following form:

$$\ddot{\varepsilon} + \left[\frac{q - GR}{MR^2} - \frac{2\Phi(\omega t)}{MR}\right]\varepsilon = 0.$$

If, moreover, the perturbing function is harmonic

$$\Phi\ (\omega t) = P_0\ \cos\ \omega t,$$

then the equation of motion is reduced to a homogeneous Mathieu equation

$$\ddot{\varepsilon} + \left[\frac{q - GR}{MR^2} - \frac{2P_0}{MP} \cos \omega t \right] \varepsilon = 0.$$

It is not hard to notice, that the mechanism portrayed in Figure 4.5 when $\alpha_0 = 0$ represents a simplified model of the known problem on the behavior of an elastic rod loaded with a periodic longitudinal force [7].

Later, the ideas on which this work is based, were developed by a number of Soviet (see, for example [6, 81], etc.,) and foreign (see, for example [95, 106]) scientists.

In our example, the distribution of the mass and elasticity of the rod are replaced with concentrated values, as the result of which the system possesses one degree of mobility.

§ 4.4. Motion of the Mechanism during the Vibration of the Stand.

A relatively large number of works has been devoted to the problems of the effect of vibration on the behavior of mechanical systems. In work [114] it was demonstrated theoretically for the first time that it is possible to ensure the stability of a free vertically-arranged rod by applying to its lower end a periodic force, acting in the vertical direction, coincident with /128
the axis of the rod. In the subsequent years a number of works appeared in which the effect of vibration of the suspension point on the motion of a mathematical or physical pendulum was studied. In some of them (see [58, 88, 92]) only a descriptive characteristic of this motion is given. In other works (see [89, 95, 96, 97]) an analysis of small oscillations of the pendulum with different character of motion of the suspension point has been performed, and the regions of its stable positions have been determined. Finally, in work [100] an examination was made of the conditions of a pendulum's motion in the process of its passing from the position of unstable equilibrium into the dynamically stable position.

In general, the range of problems arising in the study of various systems of motion of even such a relatively-simple system as the common pendulum, is extremely broad, however, not always by far, can the methods of solution of these problems and the results thus obtained, be applied to more complex systems. In the next chapter we shall examine this problem in greater detail, while now we shall set up the equations of the forced oscillations of a mechanism operating under the conditions of a vibrating stand.

Let all the forces applied to the components of the mechanism pertain to a group of potential forces or be forces which are changing slowly in the time function. Then, when the stand is motionless, the mechanism will either remain motionless or its position of equilibrium will change smoothly with the passage of time.

If in the process of operation the body of the instrument or subassembly of a machine with which the stand of the mechanism is connected, vibrates, then these vibrations are fully or partially transmitted to the stand of the mechanism even if the latter is coupled with the vibrating object by some shock absorbers, as it is depicted conventionally in Figure 4.6.

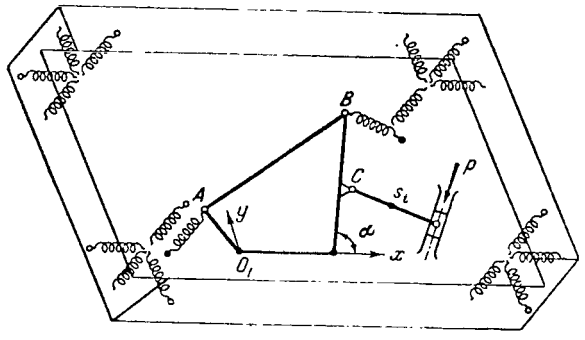

Figure 4.6.

In the general case the stand of the mechanism may have six degrees of mobility, as the result of which the picture of its vibrations will be extremely complex.

As applied to the flat mechanisms examined by us, the greater effect on their motion is exercised by the stand's vibrations, occurring in the plane parallel to the plane of motion of its components. Therefore, in the future we shall /129 disregard the effect of other vibrations occurring in planes perpendicular to the plane of the mechanism.

We shall consider that the parameters of the stand's vibrations in the direction of x and y axes (Figure 4.6) located in the plane of the mechanism, and also with respect to the axis perpendicular to the above-mentioned plane, are prescribed. Then we can proceed to set up of the equation of motion of the mechanism, assuming that the oscillations are performed with respect to a certain position of dynamic equilibrium. As before, we take into account the action of only one external force P.

Let us examine one by one all the terms included in the Lagrangian equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varepsilon}}\right)-\frac{\partial T}{\partial \varepsilon}=P\frac{ds_P}{d\varepsilon}-\frac{d\Pi}{d\varepsilon}. \qquad (4.24)$$

Let us begin with term $P\frac{ds_P}{d\alpha}$. Earlier it was specified that during the time period examined the external force P will either remain constant or will change so slowly that we can disregard its dynamic effect. The value of derivative $\frac{dsp}{d\varepsilon}$ does not depend on the position of the stand. Consequently, component $P(ds_p/d\varepsilon)$ of the generalized moment remains the same as if the mechanism moved with respect to the immobile stand, having been taken out of /130 the position of dynamic equilibrium.

Now, let us proceed with the determination of component $\frac{d\Pi}{d\varepsilon}$. The

potential energy of deformation of elastic components depends only on the position of the mechanism's components with respect to the stand. As for the forces of gravity of the components, in the process of the vibration of the stand, their potential energy varies periodically. These variations correspond to the variation of the mechanism's position with respect to the initial level, from which begins the count of the potential energy of both forces.

In this way, the component of the potential energy of the forces of gravity is equal to

$$\Pi_t = \sum_0^n G_i y_i + \Pi_0(t).$$

However, equation (4.24) includes the derivative of the potential energy. Since $\Pi_0(t)$ does not depend on ε,

$$\frac{d\Pi_0(t)}{d\varepsilon} = 0.$$

Consequently, the term $d\Pi/d\varepsilon$ also does not change and the right member of equation (4.24) has the following appearance:

$$M = D - K_{dyn}\varepsilon. \tag{4.25}$$

Let us now examine the terms contained in the left member of equation (4.24), for which we determine the kinetic energy of the mechanism, the value of which in the process of the stand's vibration undergoes substantial changes. Designating the linear and angular shifts of the stand with respect to the motionless system of coordinates with x, y and ϕ, and keeping in mind that these movements are certain functions of time $x = x(t)$, $y = y(t)$, $\phi = \phi(t)$, we get

$$\frac{dx}{dt} = \dot{x}(t), \quad \frac{dy}{dt} = \dot{y}(t), \quad \frac{d\varphi}{dt} = \dot{\varphi}(t).$$

Under the conditions of vibration of the stand of the mechanism, the absolute (angular and linear) velocities of the components and their points are the geometrical sums of the relative* and translational** velocities respective-/131
ly.

The relative angular and linear velocities of the mobile components of the mechanism performing small oscillations about the position of dynamic equilibrium, by analogy with (4.12) may be presented in the following form:

* With respect to the stand.
** Together with the stand.

$$\omega_i = \frac{d\alpha_i}{dt} = \left(\frac{d\alpha_i}{d\alpha}\right)_{\text{dyn}} \dot{\varepsilon} + \ldots,$$
$$v_{si} = \frac{d\bar{s}_i}{dt} = \left(\frac{d\bar{s}_i}{d\alpha}\right)_{\text{dyn}} \dot{\varepsilon} + \ldots,$$

where v_{si} is the velocity of the center of gravity of the i-th component with respect to the stand.

The full kinetic energy of the i-th component of the mechanism is

$$T_i = T_{iI} + T_{iII} = J_i \frac{\Omega_i^2}{2} + m_i \frac{V_{si}^2}{2},$$

where

$$\Omega_i = \left(\frac{d\alpha_i}{d\alpha}\right)_{\text{dyn}} \dot{\varepsilon} + \dot{\varphi},$$
$$\bar{V}_{si} = \bar{v}_{si} + \dot{\bar{x}} + \dot{\bar{y}}.$$

We calculate the left member of the Lagrangian equation for component T_{iI} of the kinetic energy of the i-th component, which is equal to

$$T_{iI} = J_i \frac{\Omega_i^2}{2} = \frac{J_i}{2}\left[\left(\frac{d\alpha_i}{d\alpha}\right)^2_{\text{dyn}} \dot{\varepsilon}^2 + \dot{\varphi}^2 + 2\left(\frac{d\alpha_i}{d\alpha}\right)_{\text{dyn}} \dot{\varepsilon}\dot{\varphi}\right].$$

Differentiating with respect to ε, we get

$$\frac{\partial T_{iI}}{\partial \varepsilon} = 0.$$

As the result of differentiation of T_{iI} with respect to $\dot{\varepsilon}$ we get

$$\frac{\partial T_{iI}}{\partial \dot{\varepsilon}} = J_i\left[\left(\frac{d\alpha_i}{d\alpha}\right)^2_{\text{dyn}} \dot{\varepsilon} + \left(\frac{d\alpha_i}{d\alpha}\right)^2_{\text{dyn}} \dot{\varphi}\right].$$

The subsequent differentiation with respect to t gives us /132

$$\frac{d}{dt}\left(\frac{\partial T_{iI}}{\partial \dot{\varepsilon}}\right) = J_i\left(\frac{d\alpha_i}{d\alpha}\right)^2_{\text{dyn}} \ddot{\varepsilon} + J_i\left(\frac{d\alpha_i}{d\alpha}\right)_{\text{dyn}} \ddot{\varphi}.$$

Adding up n components at a time we get for the entire mechanism

$$\frac{\partial T}{\partial \varepsilon} = 0,$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varepsilon}}\right) = \ddot{\varepsilon}\sum_{i=1}^{n} J_i\left(\frac{d\alpha_i}{d\alpha}\right)^2_{\text{dyn}} + \ddot{\varphi}\sum_{i=1}^{n} J_i\left(\frac{d\alpha_i}{d\alpha}\right)_{\text{dyn}}$$

Let us now turn to the calculation of component T_{iII} of the kinetic energy of the i-th component.

In the forward motion to every point of any of the components of the mechanism, the same translation velocity is imparted, the value of which is determined by the value of components $\dot{x}$ and $\dot{y}$. Instead of taking these components into account one by one, we add them up and perform a calculation, assuming that to the entire mechanism a translation velocity is imparted, the velocity of which is equal to

$$u = \sqrt{\dot{x}^2 + \dot{y}^2},$$

the direction of vector $\bar{u}$ of the velocity varying with time according to a specific law. Then the absolute velocity of the center of gravity of the i-th component will be equal to the geometrical sum

$$\bar{V}_{si} = \bar{v}_{si} + \bar{u}.$$

Let us build the plan of the velocities of the mechanism (see Figure 4.7) with an immobile stand. The angle formed by the direction of the vector of the velocity of the center of gravity of the i-th component and direction of axis x, will be designated with α_{si}. It goes without saying that, in case of the vibration of the stand, this will be the plan of the relative and not the absolute velocities. /133

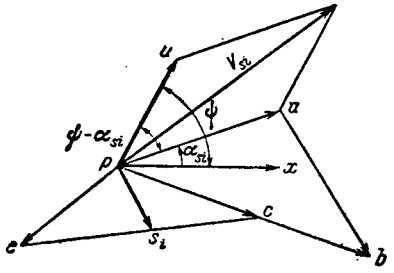

Figure 4.7.

The absolute velocity of any point of the mechanism, including the velocity of the center of gravity of the i-th component, may be found by applying the vector of translation velocity in the pole of the plane of velocities The instantaneous value of the angle formed by vector $\bar{u}$ with the vector portraying the relative velocity of the center of gravity of the i-th component, is equal to $(\psi - \alpha_{si})$. With the known vibration parameters in the direction of axes x and y, angle ψ is a certain time function.

Now, knowing the value of the absolute velocity of the center of gravity of the i-th component V_{si}, we can write the kinetic energy T_{iII} of this component in the following way:

$$T_{iII} = m_i \frac{V_{si}^2}{2} = \frac{m_i}{2}\left[\left(\frac{ds_i}{d\alpha}\right)^2_{\text{dyn}} \dot{\varepsilon}^2 + u^2 + 2\left(\frac{ds_i}{d\alpha}\right)_{\text{dyn}} \dot{\varepsilon} u \cos(\psi - \alpha_{si})\right].$$

In the process of small oscillations of the mechanism with respect to the stand, angle α_{si} changes through the change of the mechanism's shape. The value of this angle may be presented in the form of the following sum:

$$\alpha_{si} = \alpha_{si\ dyn} + \varepsilon_i,$$

where $\alpha_{si\ dyn}$ is the value corresponding to the position of the dynamic equilibrium of the mechanism, and $\varepsilon_i \simeq (d\alpha_i/d\alpha)_{dyn}\ \varepsilon$ is the value of the same order of smallness as value ε*. Disregarding the values of the second order of smallness, and assuming that $\sin \varepsilon = \varepsilon$, $\cos \varepsilon = 1$, we present the functions $\cos \alpha_{si}$ and $\sin \alpha_{si}$ in the following form:

$$\cos \alpha_{si} = \cos (\alpha_{si\ dyn} + \varepsilon_i) = \cos \alpha_{si\ dyn} - \left(\frac{d\alpha_i}{d\alpha}\right)_{dyn} \varepsilon \sin \alpha_{si\ dyn},$$

$$\sin \alpha_{si} = \sin (\alpha_{si\ dyn} + \varepsilon_i) = \sin \alpha_{si\ dyn} + \left(\frac{d\alpha_i}{d\alpha}\right)_{dyn} \varepsilon \cos \alpha_{si\ dyn}.$$

Bearing in mind that /134

$$\cos (\psi - \alpha_{si}) = \cos \psi \cos \alpha_{si} + \sin \psi \sin \alpha_{si},$$

after transformations we get

$$\cos (\psi - \alpha_{si}) = \cos (\psi - \alpha_{si\ dyn}) - \left(\frac{d\alpha_i}{d\alpha}\right)_{dyn} \varepsilon \sin (\psi - \alpha_{si\ dyn})$$

As the result the expression of the component of kinetic energy T_{iII} assumes the following appearance:

$$T_{iII} = \frac{m_i}{2}\left\{\left(\frac{ds_i}{d\alpha}\right)^2_{dyn} \dot{\varepsilon}^2 + u^2 + 2\left(\frac{ds_i}{d\alpha}\right)_{dyn} \dot{\varepsilon} u \left[\cos (\psi - \alpha_{si\ dyn}) - \left(\frac{d\alpha_i}{d\alpha}\right)_{dyn} \varepsilon \sin (\psi - \alpha_{si\ dyn})\right]\right\}.$$

After differentiation with resepct to ε, $\dot{\varepsilon}$, and t, and totaling n components at a time, the left member of the Lagrangian equation of component T_{iII} assumes the following appearance:

* We exclude from examination certain special positions of the mechanism, for which this statement loses its value (see for example [41]).

$$\frac{d}{dt}\left(\frac{\partial T_{II}}{\partial \dot{\varepsilon}}\right)-\frac{\partial T_{II}}{\partial \varepsilon}=\ddot{\varepsilon}\sum_{i=0}^{n}m_i\left(\frac{ds_i}{d\alpha}\right)^2_{\text{dyn}}-$$
$$-\varepsilon\sum_{0}^{n}m_i\left(\frac{ds_i}{d\alpha}\right)_{\text{dyn}}\left(\frac{d\alpha_i}{d\alpha}\right)_{\text{dyn}}[\dot{u}\sin(\psi-\alpha_{si\,\text{dyn}})+$$
$$+u\dot{\psi}\cos(\psi-\alpha_{si\,\text{dyn}})]+$$
$$+\sum_{i=0}^{n}m_i\left(\frac{ds_i}{d\alpha}\right)_{\text{dyn}}[\dot{u}\cos(\psi-\alpha_{si\,\text{dyn}})+u\dot{\psi}\sin(\psi-\alpha_{si\,\text{dyn}})]. \tag{4.27}$$

Substituting (4.25), (4.26), and (4.27) in (4.24) and designating:

$$\frac{1}{J_{\text{dyn}}}\sum_{i=0}^{n}m_i\left(\frac{ds_i}{d\alpha}\right)_{\text{dyn}}\left(\frac{d\alpha_i}{d\alpha}\right)_{\text{dyn}}[\dot{u}\sin(\psi-\alpha_{si\,\text{dyn}})+u\dot{\psi}\cos(\psi-\alpha_{si\,\text{dyn}})]=\Psi(t). \tag{4.28}$$

$$\frac{1}{J_{\text{dyn}}}\sum_{i=0}^{n}m_i\left(\frac{ds_i}{d\alpha}\right)_{\text{dyn}}[\dot{u}\cos(\psi-\alpha_{si\,\text{dyn}})- \qquad \text{/135}$$
$$-u\dot{\psi}\sin(\psi-\alpha_{si\,\text{dyn}})+\frac{\ddot{\varphi}}{J_{\text{dyn}}}\sum_{i=0}^{n}J_i\left(\frac{d\alpha_i}{d\alpha}\right)_{\text{dyn}}=\Theta(t), \tag{4.29}$$

we obtain the following equation of motion of the mechanism:

$$\ddot{\varepsilon}+[\omega^2_{n\,\text{dyn}}-\Psi(t)]\,\varepsilon=d-\Theta(t). \tag{4.30}$$

Here, functions $\Psi(t)$ and $\Theta(t)$ may be periodic or aperiodic depending on whether actuation conditions are settled or unsettled.

We are examining only the settled actuation conditions. In this case the motion of the mechanism operating under the conditions of plane vibration of the stand satisfies the inhomogeneous differential equation of the second order with periodic coefficients.

Let us examine some particular cases, proceeding from the analysis given above:

a) The stand vibrates only with respect to the axis perpendicular to the plane of the mechanism. Vibrations along axes x and y are absent, and consequently

$$\dot{u} = u = 0.$$

In the future such a vibration will be called a torsion vibration. For a

torsion vibration functions $\Psi(t)$ and $\Theta(t)$, according to expressions (4.28) and (4.29) will be equal to

$$\Psi(t)=0;\ \Theta(t)=-\frac{\ddot{\varphi}}{J_{\text{dyn}}}\sum_{i=0}^{n}J_i\left(\frac{d\alpha_i}{d\alpha}\right)_{\text{dyn}}=\Theta_1\ddot{\varphi},$$

while equation (4.30) assumes the appearance of

$$\ddot{\varepsilon}+\omega_{n\,\text{dyn}}^2\varepsilon=d-\Theta_1\ddot{\varphi}.$$

Thus, in the case of a torsion vibration the motion of the mechanism is determined by the usual linear inhomogeneous differential equation with constant coefficients. Coefficient Θ_1 contained in the right member of the equation will be called the actuation coefficient. Its value is determined by the parameters /136 of the mechanism.

If the periodic actuation function is such that its mean value for the period is equal to zero

$$\frac{1}{T}\int_0^T\varphi\,dt=0,$$

then the constant term d is equivalently equal to zero and the equation of motion assumes the following appearance:

$$\ddot{\varepsilon}+\omega_n^2\varepsilon=\Theta_1\ddot{\varphi}. \tag{4.31}$$

Here, the forced oscillations of the mechanism will occur around the position of static equilibrium of the mechanism.

If the torsion vibrations satisfy the harmonic law

$$\phi = \phi_0 \cos \omega t,$$

where ω is the angular velocity, and ϕ_0 is the amplitude of the stand's vibration, then the equation of motion of the mechanism assumes the following appearance:

$$\ddot{\varepsilon}+\omega_n^2\varepsilon=-\varphi_0\Theta_1\omega^2\cos\omega t.$$

Considering that the free oscillations attenuate with time, the solution of this equation will be written in the following form:

$$\varepsilon=\frac{\varphi_0\Theta_1}{1-\left(\frac{\omega_n}{\omega}\right)^2}\cos\omega t.$$

Consequently, with harmonic torsion vibrations of the stand, the mechanism, with respect to the position of static equilibrium, performs harmonic vibrations, the amplitude of which depends on the parameters of the vibration and the value of the actuation coefficient.

b) The translation vibration of the stand in a constant direction. The case of the reciprocating vibration of the stand is the most important one. The theoretical analysis of this case enables us to determine the characteristic features of the effect of vibration on the behavior of the mechanism with elastic couplings. In addition to this, all the experimental investigations /137 and acceptance tests of instruments and machine subassemblies, subjected to vibration action in the course of operation, are performed, as a rule, precisely under the conditions of reciprocating vibration, having a harmonic character. Owing to the above-mentioned circumstances, the analysis of the action of this type of actuation is given the greatest attention in theoretical and experimental investigations of the dynamic and stability of mechanisms, machines, and instruments. With reciprocating vibration of the stand, function $\phi(t) \equiv 0$, and the vector of the translation velocity, while it varies with respect to value, remains constant with respect to direction, so that

$$\psi = \psi_0 = \text{const}, \quad \dot{\psi} = 0.$$

Having substituted these values in (4.28) and (4.29), we get

$$\Psi(t) = \frac{1}{J_{\text{dyn}}} \sum_{i=0}^{n} m_i \left(\frac{ds_i}{d\alpha}\right)_{\text{dyn}} \left(\frac{d\alpha_i}{d\alpha}\right)_{\text{dyn}} \dot{u} \sin(\psi_0 - \alpha_{si\,\text{dyn}}) = \Psi_2 \dot{u}$$

$$\Theta(t) = \frac{1}{J_{\text{dyn}}} \sum_{i=0}^{n} m_i \left(\frac{ds_i}{d\alpha}\right)_{\text{dyn}} \dot{u} \cos(\psi_0 - \alpha_{si\,\text{dyn}}) = \Theta_2 \dot{u}.$$

Accordingly, equation (4.30) assumes the following appearance:

$$\ddot{\varepsilon} + (\omega^2_{n\,\text{dyn}} + \Psi_2 \dot{u})\,\varepsilon = d + \Theta_2 \dot{u}. \tag{4.32}$$

Thus, in the case of translation vibration the motion of the mechanism is determined by a linear differential equation with a periodic coefficient.

We shall call value

$$\Psi_2 = \frac{1}{J_{\text{dyn}}} \sum_{i=0}^{n} m_i \left(\frac{ds_i}{d\alpha}\right)_{\text{dyn}} \left(\frac{d\alpha_i}{d\alpha}\right)_{\text{dyn}} \sin(\psi_0 - \alpha_{si\,\text{dyn}})$$

the periodicity coefficient. Its presence is inalineably connected with the presence in the kinematic chain of components, performing a relatively stable complex motion or rotating around axes which do not correspond with their centers of gravity. If the mechanism's components move in the forward direction/138

with respect to the stand or are rotating about axes coincident with the centers of gravity, the coefficient of periodicity is identically equal to zero (since in this case either $(d\alpha_i/d\alpha)_{dyn}$ or $(ds_i/d\alpha)_{dyn} = 0$). The value

$$\Theta_2 = \frac{1}{J_{dyn}} \sum_{i=0}^{n} m_i \left(\frac{ds_i}{d\alpha}\right)_{dyn} \cos(\psi_0 - \alpha_{si\,dyn})$$

represents the coefficient of actuation. Its value is determined by the presence in the kinematic chain of the mechanism of components the centers of gravity of which in the process of oscillations move with respect to the stand.

It is not hard to see that the mechanism which consists only of components rotating about the centers of gravity (an ideally-balanced mechanism), is not affected at all by the translation vibrations of the stand, since in this case both Ψ_2 and Θ_2 are identically equal to zero.

Comparing (4.31) and (4.32), we can establish the basic difference in the behavior of the mechanism of the torsion and reciprocating vibrations of the stand. It is determined by the presence in the left-member of equation (4.32) of a periodic member (Ψ_2 u ε), which may have a substantial effect on the behavior of the mechanism.

If the translation vibrations are performed according to the harmonic law

$$\rho = \rho_0 \cos \omega t,$$

then equation (4.32) assumes the following appearance

$$\ddot{\varepsilon} + (\omega_{n\,dyn}^2 + \varrho_0 \omega^2 \Psi_2 \cos \omega t)\,\varepsilon = d + \varrho_0 \omega^2 \Theta_2 \cos \omega t. \qquad (4.33)$$

Comparing equations (4.22) and (4.32) and (4.23) and (4.33) respectively, we can discover that the action of reciprocating vibrations, is similar to the action of a certain pulsating force, applied to one of the mobile components of the mechanism.

This similarity becomes comprehensible if we take into account that the vibrations of the stand occur practically under the action of a system of forces which can be reduced to the resulting moment and the resulting force, acting upon the stand, which is one of the components of the mechanism. /139

Example. Let us go back to the mechanism presented in Figure 4.5, and assume that this mechanism is fastened on a mobile stand, performing in the vertical direction a periodic motion with respect to the immobile guide rails (Figure 4.8).

The static equilibrium position in the case of an immobile stand, is still determined from the equation

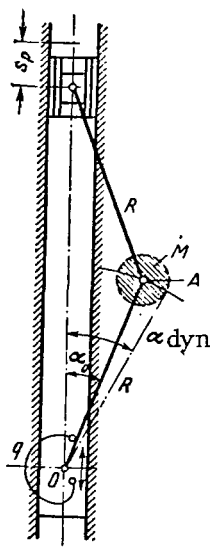

Figure 4.8.

$$\alpha_0 = \frac{GR}{q}\sin\alpha_0,$$

whereas the mechanism oscillations' own frequency in this position is equal to

$$\omega_n = \sqrt{\frac{q - GR\cos\alpha_0}{MR^2}}.$$

Let us form, using (4.32) the equation of motion of a mechanism with respect to the vibrating stand. For this it suffices to determine for the example examined the values of the coefficients of periodicity and actuation.

Let us construct a plan of velocities of the mechanism (Figure 4.9) from which it follows that

$$\sin(\psi_0 - \alpha_{si_{dyn}}) = \sin\left(\frac{\pi}{2} \pm \alpha_{dyn}\right) = \cos\alpha_{dyn}.$$

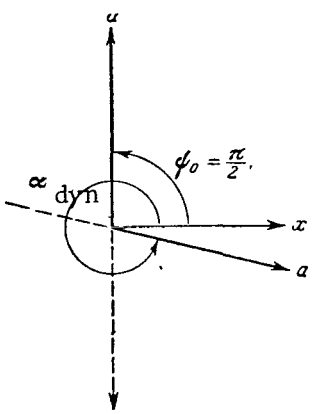

Figure 4.9.

The derivatives included in the periodicity coefficient are

$$\left(\frac{ds_i}{d\alpha}\right) = \left(\frac{ds_A}{d\alpha}\right) = R = \text{const},$$
$$\left(\frac{d\alpha_i}{d\alpha}\right) = 1.$$

The inertial coefficient, as before, is equal to

$$J_{dyn} = MR^2.$$

Thus, the periodicity coefficient will be /140

$$\Psi_2 = \frac{\cos\alpha_{dyn}}{R}.$$

Let us set up the equation for the actuation coefficient. According to the velocity plan

$$\cos(\psi_0 - \alpha_{si_{dyn}}) = \cos\left(\frac{\pi}{2} \pm \alpha_{dyn}\right) = \mp\sin\alpha_{dyn}.$$

The sign of the actuation coefficient depends on the relationship between the frequency of free oscillations ω_n and the frequency of vibration ω. When $\omega_n < \omega$, $\Theta_2 < 0$, owing to the fact that the phases of motion of a mobile stand

and slider are shifted with respect to one another by π. The velocity plan in Figure 4.9 is plotted precisely for such an occasion.

Finally, the constant component of the generalized moment proves equal to

$$d = \frac{GR \sin \alpha_{dyn} - q\alpha_{dyn}}{MR^2}.$$

Now, we may write the equation of forced oscillations of the mechanism with respect to the vibrating stand

$$\ddot{\varepsilon} + \left[\frac{q - GR\cos\alpha_{dyn}}{MR^2} - \frac{\cos\alpha_{dyn}}{R}\dot{u}\right]\varepsilon = \frac{GR\sin\alpha_{dyn} - q\alpha_{dyn}}{MR^2} + \frac{\sin\alpha_{dyn}}{R}\dot{u}$$

If the vibration of the stand is performed according to the harmonic law, this equation assumes the following appearance:

$$\ddot{\varepsilon} + \left[\frac{q - GR\cos\alpha_{dyn}}{MR^2} + \frac{\varrho_0\omega^2\cos\alpha_{dyn}}{R}\cos\omega t\right]\varepsilon = \frac{GR\sin\alpha_{dyn} - q\alpha_{dyn}}{MR^2} - \frac{\varrho_0\omega^2\sin\alpha_{dyn}}{R}\cos\omega t.$$

Both equations, as it should have been expected, are identical to the equations set up on § 4.3 for the case of the action of a pulsating force on the mechanism examined. /141

c) The translation circular vibration of the stand. We examined the case when the components of the translation vibration along axes x and y, upon being added, produce the vector, constant in direction and variable in value. Here, the stand of the mechanism performs reciprocating motions.

Later on we shall encounter the inverse instance, namely one, when the components of the translating vibration, on being added up, yield a vector constant in value and rotating uniformly. In this case the mechanism's stand will perform a translation motion along the circumference.

In the process of the translation-circular vibrations, every point of the mechanism in the translation motion describes a circle, and consequently,

$$u = \varrho_0\omega, \quad \psi = \omega t, \quad \omega = \text{const}.$$

Turning to equations (4.28) and (4.29), we can see that under the above-mentioned conditions the coefficient of periodicity and coefficient of actuation in equation (4.30) will be respectively equal to

$$\Psi(t) = \frac{\varrho_0\omega^2}{J_{dyn}}\sum_{i=0}^{n} m_i\left(\frac{ds_i}{d\alpha}\right)_{dyn}\left(\frac{d\alpha_i}{d\alpha}\right)_{dyn}\cos(\omega t + \psi_0 - \alpha_{si\,dyn}),$$

$$\Theta(t) = -\frac{\varrho_0\omega^2}{J_{dyn}}\sum_{i=0}^{n} m_i\left(\frac{ds_i}{d\alpha}\right)_{dyn}\sin(\omega t + \psi_0 - \alpha_{si\,dyn}).$$

Adding up the individual harmonics, found under the signs of the sums, and selecting phase ψ_0 in the corresponding manner, we obtain the following expressions:

$$\Psi(t) = \frac{\varrho_0\omega^2}{J_{\text{dyn}}} R_\Psi \cos\omega t,$$
$$\Theta(t) = \frac{\varrho_0\omega^2}{J_{\text{dyn}}} R_\Theta \sin(\omega t + A).$$

Consequently, the equation of motion of a mechanism, the stand of which performs a translation-circular vibration, has the following appearance: /142

$$\ddot{\varepsilon} + \left[\omega_n^2 \text{ dyn} - \frac{\varrho_0\omega^2}{J_{\text{dyn}}} R_\Psi \cos\omega t\right]\varepsilon = d + \frac{\varrho_0\omega^2}{J_{\text{dyn}}} R_\Theta \sin(\omega t + A). \qquad (4.34)$$

This equation, as did equation (4.33) is an inhomogeneous Mathieu equation; it is not hard to see that a translation-circular vibration has no effect whatever on an ideally-balanced mechanism, since in this case $\Psi(t) = \Theta(t) \equiv 0$, owing to which for each of components $(d_{si}/d\alpha) \equiv 0$.

However, this conclusion is correct only under the condition of the absence of friction in the kinematic pairs, with which the mechanism is coupled to the vibrating stand.

§ 4.5. Equation of Motion in the Canonical Form.

For the convenience of the further analysis we shall make a summation of the differential equations of the motion of mechanisms with elastic couplings obtained in this chapter (see [40]).

In the case of the action of a pulsing force the equation of small oscillations of the mechanism was obtained in the form of (4.22)

$$\ddot{\varepsilon} + \left[\omega_n^2 \text{ dyn} \left(\frac{d^2 s_P}{d\alpha^2}\right)_{\text{dyn}} \frac{\Phi(\omega t)}{J_{\text{dyn}}}\right]\varepsilon = d + \left(\frac{ds_P}{d\alpha}\right)_{\text{dyn}} \frac{\Phi(\omega t)}{J_{\text{dyn}}}.$$

If the pulsation of the force obeys the harmonic law, this equation assumes the form of (4.23)

$$\ddot{\varepsilon} + \left[\omega_n^2 \text{ dyn} \left(\frac{d^2 s_P}{d\alpha^2}\right)_{\text{dyn}} \frac{P_0}{J_{\text{dyn}}} \cos\omega t\right]\varepsilon = d + \frac{P_0}{J_{\text{dyn}}} \left(\frac{ds_P}{d\alpha}\right)_{\text{dyn}} \cos\omega t.$$

The motion of the mechanism in the process of a plane reciprocal vibration of the stand satisfies equation (4.32)

$$\ddot{\varepsilon} + [\omega_{n\,\text{dyn}}^2 + \Psi_2 \dot{u}]\,\varepsilon = d + \Theta_2 \dot{u},$$

where

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$$\dot{u} = \dot{u}(t).$$

If the vibration of the stand obeys the harmonic law, this equation assumes the appearance of (4.33)

$$\ddot{\varepsilon} + [\omega_{n\,\text{dyn}}^2 + \varrho_0 \omega^2 \Psi_2 \cos \omega t]\,\varepsilon = d + \varrho_0 \omega^2 \Theta_2 \cos \omega t.$$

We designate

$$\omega t = 2x, \quad \varepsilon = y.$$

Then, bearing in mind that $\varepsilon = \varepsilon(\omega t)$, and therefore $\ddot{\varepsilon} = \frac{\omega^2}{4} \cdot \frac{d^2y}{dx^2}$, we can rewrite the four above-adduced equations in the following way:

$$\frac{d^2y}{dx^2} + \left[\frac{4\omega_n^2}{\omega^2}\text{dyn} - \frac{4\,\Phi(2x)}{\omega^2 J_{\text{dyn}}}\left(\frac{d^2s_P}{dx^2}\right)_{\text{dyn}}\right] y = \frac{4d}{\omega^2} + \frac{4\,\Phi(2x)}{\omega^2 J_{\text{dyn}}}\left(\frac{ds_P}{dx}\right)_{\text{dyn}}, \tag{4.35}$$

$$\frac{d^2y}{dx^2} + \left[\frac{4\omega_n^2}{\omega^2}\text{dyn} - \frac{4\,\varrho_0}{\omega^2 J_{\text{dyn}}}\left(\frac{d^2s_P}{dx^2}\right)_{\text{dyn}} \cos 2x\right] y = \frac{4d}{\omega^2} + \frac{4\,\varrho_0}{\omega^2 J_{\text{dyn}}}\left(\frac{ds_P}{dx}\right)_{\text{dyn}} \cos 2x, \tag{4.36}$$

$$\frac{d^2y}{dx^2} + \left[\frac{4\omega_n^2}{\omega^2}\text{dyn} - \frac{4\Psi_2}{\omega^2} u'\right] y = \frac{4d}{\omega^2} - \frac{4\Theta_2}{\omega^2} u', \tag{4.37}$$

$$\frac{d^2y}{dx^2} + \left[\frac{4\omega_n^2}{\omega^2}\text{dyn} + 4\varrho_0 \Psi_2 \cos 2x\right] y = \frac{4d}{\omega^2} + 4\varrho_0 \Theta_2 \cos 2x, \tag{4.38}$$

or, in the general form

$$\frac{d^2y}{dx^2} + q(x)\,y = f(x). \tag{4.39}$$

Functions q(x) and f(x) are determined by the character of the actuation. In the case of its periodicity they satisfy the conditions

$$q(x + \pi) = q(x), \quad f(x + \pi) = f(x)$$

and may be expanded into the Fourier series.

The reduction of the equations of motion to the canonical form will be done using the following transformations. We designate the mean value of function q(x) with a: /144

$$a = \frac{1}{\pi}\int_0^{\pi} q(x)\,dx.$$

We designate the modulus of the maximum value of difference a - q(x) with 2q, i.e., $|a - q(x)|_{max} = 2q$. Now function q(x) may be written in the following way:

$$q(x) = a - 2q \cdot g(x), \tag{4.40}$$

where a and q are constant values, and g(x) is a periodical function, having the following properties:

$$g(x+\pi) = g(x), \quad \int_0^{\pi} g(x)\,dx = 0, \quad |g(x)| < 1.$$

We present function g(x) in the form of a Fourier series, having written this series in the complex form:

$$g(x) = \sum_{r=-\infty}^{\infty}{}^{*} q_{2r} e^{2rix} \tag{4.41}$$

(the asterisk signifies that the sum does not contain the term r = 0).

We remind the reader that if function g(x) is not even, and not odd, then coefficients q_{2r} represent complex values, while q_{2r} and q_{-2r} are conjugated in such a way that $q_{-2r} = \bar{q}_{2r}$, where $\bar{q}_{2r}$ is a number conjugated with number q_{2r}.

Substituting in equation (4.39) expression q(x), set up according to equations (4.40) and (4.41), we get a Hill's equation in the form of

$$\frac{d^2y}{dx^2} + \left(a - \sum_{i=-\infty}^{\infty}{}^{*} q_{2r} e^{2rix}\right) y = f(x). \tag{4.42}$$

If function g(x) is even, then coefficients q_{2r} represent real numbers. If, moreover, we make all coefficients q_{2r} equal to zero, with the expression of q_2 (when r = 1), we obtain a Mathieu equation which has the following appearance

$$\frac{d^2y}{dx^2} + (a - 2q\cos 2x)\, y = f(x). \tag{4.43}$$

Thus, having expanded the actuation function into a Fourier series, and determined the coefficients of this series we can always, by means of the above-mentioned transformation, reduce the equation of motion to the (4.42) form. /145

In actual practice, as it had been mentioned in the introduction, it is attempted to separate the main harmonic components from the entire spectrum of excitation harmonics. When we take into consideration only these components, the equation of motion assumes the appearance of (4.43).

Now we return to the above-adduced equations and introduce the following designations:

$$\left(\frac{2\omega_n \text{dyn}}{\omega}\right)^2 = a, \tag{4.44}$$

$$\frac{2\left(\frac{d^2 s_P}{da^2}\right)_{\text{dyn}}}{\omega^2 J_{\text{dyn}}} P_0 = q_1, \tag{4.45}$$

$$-2\varrho_0 \Psi_2 = q_2, \tag{4.46}$$

$$\frac{4d}{\omega^2} = Q_0, \tag{4.47}$$

$$\frac{4\left(\frac{ds_P}{da}\right)_{\text{dyn}}}{\omega^2 J_{\text{dyn}}} P_0 = Q_1, \tag{4.48}$$

$$4\varrho_0 \Theta_2 = Q_2. \tag{4.49}$$

Using these designations, we can write equations (4.36) and (4.38) in the following form:

$$\frac{d^2 y}{dx^2} + (a - 2q_j \cos 2x)\, y = Q_0 + Q_j \cos 2x; \quad j = 1, 2. \tag{4.50}$$

By means of the above-mentioned transformation, equations (4.35) and (4.37) are reduced to the canonic type of the inhomogeneous Hill's equation, however, we shall not carry out a more detailed analysis of cases leading to this equation.

Having written the equations of motion in this form, in the future, we shall find it easy to apply to their analysis the methods which were explained in sufficient detail in the second chapter.

CHAPTER 5

DYNAMIC EQUILIBRIUM OF A MECHANISM WITH ELASTIC COUPLINGS

§ 5.1. Dynamic Precision of a Mechanism /146

Having set up the equation of motion of a mechanism with elastic couplings we can begin the solution of certain engineering problems, the essence of which is determined by the functional purpose of the mechanism. An extensive group of such problems is connected with the investigation of the dynamic precision of mechanisms.

The concept of the "dynamic precision of a mechanism" as well as the concept of the "dynamic error of a mechanism" have not been exhaustively defined up to the present time. These concepts encompass a wide range of phenomena connected with the work of inertial systems under dynamic conditions.

Naturally, first of all we have to render more precise these concepts, applicably to the problems examined in this book, which are connected, in particular, with the investigation of the dynamics of mechanisms operating under the conditions of pulsations of external forces and vibrations of individual components. We know well that pulsations and vibrations, which often originate as the unavoidable side effects accompanying the actual operating conditions, may serve as the source of substantial dynamic errors of the system.

In accordance with the above, from now on, as dynamic errors of a mechanism we will understand the additional values of its generalized coordinates and their derivatives, which arise as the result of incidental pulsations or vibrations.

In a number of branches of instrument making it is the adopted procedure to characterize the full dynamic error of an instrument by two of its components, which had been given the descriptive names withdrawal or translation, (literally: leading away), and disalignment, (literally: washout or blurring). In the future we shall use these terms widely. /147

The mechanism is led away (withdrawn) as the result of noncoincidence of its static and dynamic equilibrium. We have already foreseen the possibility of such a noncoincidence of the motions of a mechanism. In this chapter we are to determine the position of the dynamic equilibrium of the mechanism, after which it is easy to compute the withdrawal value.

The disalignment (blurring) of a mechanism is the double amplitude of forced oscillations, which the mechanism performs with respect to the position of the dynamic equilibrium. Knowing the character of the solution of the equations of equation of motion, it is easy to determine also this component

of the full dynamic error of the mechanism.

However, we should say that the equations set up in the previous chapter, do not reflect completely, by far, the factors which in the real systems provide the conditions for the origination of dynamic errors. In particular, it is well known that in the work of a system under the conditions of vibration, substantial dynamic errors may arise as the result of friction in the kinematic pairs of the mechanism. The equations of motion in the preceding chapter were set up by us without an allowance for the action of these forces.

We also know that the dynamic errors of a mechanism may be the result of the presence of clearances in the kinematic pairs, and the impact interaction between the elements of these pairs. This also has not been reflected in the equations of motion that we have set up.

Thus, examining in this chapter the problem of the dynamic precision of mechanisms on the basis of the equations set up, we shall be able to appraise the value of such important factors as the geometry of the mechanism, distribution of its masses, and characteristics of the elastic couplings, but of course, we will not exhaust this problem completely, and in the future shall return to it when we shall have to deal with the estimation of the effect of friction and in the examination of the vibro-impact conditions of motion of mechanisms with elastic couplings.

However, the analysis of the dynamic precision of a mechanism, in every instance becomes sensible only after it has been proved that the position of the dynamic equilibrium of a mechanism, established as the result of the above- /148
mentioned actions, is stable. As the statics equations enable us to determine the position of equilibrium of a system but do not answer the question, which of these positions is stable, so do the equations of dynamics also enable us to determine certain conditions of a system, but do not answer the question, which of these conditions of motion are stable. Therefore the determination of the conditions of dynamic stability of a mechanism is of interest not only as an introduction to the analysis of its dynamic precision. Ensuring the dynamic stability is necessary for the normal work of any mechanism, no matter what its functional purpose may be. The disruption of this condition, along with the formation of resonance conditions, results in an appreciable increase of the amplitude of vibrations of the individual components, increase of inertial loads, reduction of the service life of the mechanism, and has a harmful effect on the technological process carried out by this mechanism.

Bearing this in mind, in the present chapter we precede the analysis of dynamic precision by the examination of the problem of dynamic stability of the mechanism and on the conditions of origination of resonance states.

In this case the problem is reduced to the analysis of the stability of solution of equations (4.42) and (4.43), set up in the preceding chapter.

We should note that the analysis of equations of the (4.42) type representing the inhomogeneous Hill's equation, is connected with great mathematical difficulties and cumbersome computations. In order to evaluate these

difficulties it is sufficient to take into account, for example, that in investigating the problem of the stability of a solution in the presence of three parameters in the equation of motion (instead of two: a and q, contained in Mathieu's equation) we shall have to construct, if we use the geometric concepts, not a plane, but a three dimensional picture*.

Even greater difficulties arise in the analysis of these equations in connection with the investigation of dynamic precision, determination of the condition of origination of resonance conditions, etc. /149

Taking the above-mentioned difficulties into account and also bearing in mind the great practical importance of the case of monoharmonic actuation, in the future we shall limit ourselves to the analysis of the stability and dynamic precision of motion, satisfying equation (4.43).

The problems of stability, as well as the problems of elimination of the dynamic errors of the mechanisms, as applied to the simplest pendulum systems, were studied both theoretically (see, for example [92, 96]) and experimentally (see, for example [108, 116]). As for the more complex systems, such as mechanisms with elastic couplings, they were studied on the basis of the simplest kinetostatic considerations, either experimentally in the process of making final adjustments on the experimental models. Here these problems will be developed further.

After we have determined the relationships between the parameters of the mechanism and actuation parameters, which ensure its dynamic stability, we shall pass over to the determination of translation and disalignment of the mechanism with elastic couplings. Here, we shall examine in detail the important instance, enabling us to obtain a visual concept of the causes of origination of the dynamic errors, and along with it, to illustrate the method developed for the analysis of dynamic stability.

§ 5.2. Dynamic Stability of a Mechanism.

We shall call a mechanism dynamically stable in the instance when the amplitude of its oscillations remains limited in the course of time, with a sufficiently small actuation amplitude.

In the opposite case, i.e., if with a constant and a sufficiently small actuation, the amplitude of the forced oscillations of the mechanism shows a tendency for unlimited growth with time, we shall term the mechanism dynamically stable.

The solution of the problem of dynamic stability of a mechanism is connected with the investigation of the stability of solution of the homogeneous equation /150

* The author knows of only one work [98] in which an attempt is made to examine such a problem as applied to the homogeneous Hill's equation.

$$\frac{d^2y}{dx^2} + (a - 2q_j \cos 2x)\, y = 0,$$

corresponding to equation (4.50).

If parameters a and q_j (j = 1, 2) are such that this solution is stable, the amplitude of small oscillations of a mechanism under any initial conditions of motion, remains limited.

The investigation of the problems of the stability of Matheiu equation solution has been, in essence, completed. The presence of a stability "chart" (see Figure 2.1) enables us to find the answer to the question posed with the prescribed values of parameters a and q, without resorting to computations. For this it suffices to establish, into which of the zones of this chart falls the characteristic point of the mechanism, the coordinates of which are determined by the values of the mechanism's parameters and actuation parameters.

Parameters a and q (see expressions (4.44)-(4.46)) of the same mechanism may assume different values depending on the size of α_{dyn}, and also on the actuation amplitude and frequency. To solve the problem of the stability of the mechanism, it is necessary to determine here not the characteristic point alone, but an entire characteristic region. To determine this region, let us turn to the examination of the relationships connecting parameters a and q_j of the equation of motion with the parameters of the mechanism and the parameters of actuation.

Parameter a (see expression (4.44)) is a function of two variables -- α_{dyn} and ω, its value depending on the ratio between the frequency of free oscillations of the mechanism near the position of its dynamic equilibrium, and the actuation frequency. In order to establish the boundaries of the possible values of parameter a, it is not necessary to determine a number of α_{dyn} values. It is sufficient to assume that all these values are located within the limits of the prescribed range of the mechanism's work. Then the extreme values of parameter a with a fixed ω will be obtained by substituting in expression (4.44) the values $\omega_{n\,min}$ and $\omega_{n\,max}$, taken in accordance with the characteristic of the frequency of the mechanism's free oscillations (Figure 5.1).

The intermediate values of parameter a, calculated for a certain value ω, /151
in Figure 5.1 are connected by a broken line. If the excitation frequency ω does not remain constant but varies within certain limits, from ω_{min} to ω_{max}, all the possible values of the parameter are located in the region limited by lines ω_{min} and ω_{max}. In Figure 5.1 this region is shaded.

For relatively small values of q the instability of motion takes place with values a equal or close to k^2, where k = 1, 2, 3, ... (see Chapter 2).

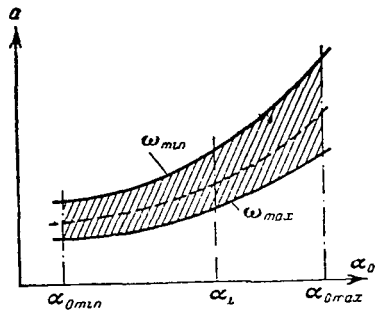

Figure 5.1.

The corresponding relationship of the frequencies according to (4.44) is equal to

$$\frac{\omega_n}{\omega}\frac{dyn}{=}\frac{k}{2}.$$

Thus, if the actuation frequency is close to value

$$\omega=\frac{2\omega_n}{k},$$

where ω_n are the values of the frequency of the mechanism's free oscillations at any point of the operating range, then there arises a danger of a substantial growth of the oscillation amplitude (parametric resonance of the mechanism) and the mechanism proves dynamically unstable. In order to determine fully the characteristic region of the mechanism's work, it is required to establish also the relationship connecting value q_j (see expressions (4.45) and (4.46)) with the parameters of the mechanism and actuation. Let us examine the case of the stand's vibration, to which corresponds $q_j = q_2 = q$. Parameter q, just as parameter a is a function of two variables α_{dyn} and ρ_0.

In Figure 5.2, a broken line shows the relationship $q = q(\alpha_{dyn})$, computed with a certain fixed value of ρ_0.

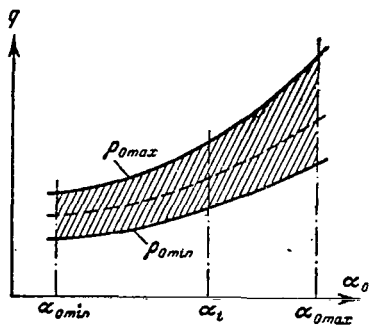

Figure 5.2.

Attributing to ρ_0 different values within the limits of $\rho_{0\,min}$ to $\rho_{0\,max}$, corresponding to the conditions of operation of the mechanism, we shall find the region of the possible values of q. In Figure 5.2 this region is shaded. In this way, in determining parameter q it is also possible to leave the question of the numerical value of α_{dyn} open, limiting ourselves to the assumption that with this value of the generalized coordinate the mechanism occupies a certain position inside its own operating range. /152

Knowing the values of a and q we can now begin the construction of the characteristic region of the mechanism. In Figure 5.3, as an example, we give that part of the stability chart which encompasses all the possible values of parameters a and q, peculiar to the mechanism under examination, and calculated in the assumption that the position of the dynamic equilibrium of the mechanism is located within the limits of the operating range.

The curves in Figures 5.1 and 5.2 enable us to plot on the stability chart

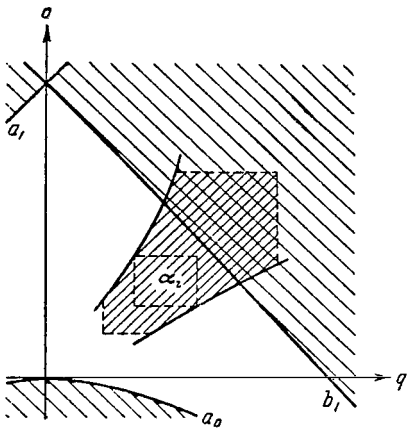

Figure 5.3.

for every value of α_i a rectangle (see Figure 5.3), which represents a particular characteristic region of the mechanism. The set of these particular regions, plotted for the entire operating range of the mechanism, forms the characteristic region in which we are interested. If a part of the characteristic regions is located in the instability zones, the construction of particular regions enables us to establish the corresponding positions of the mechanism and the vibration parameters. /153

Thus, above it was shown how to evaluate the dynamic stability of a mechanism operating under the condition of vibration of the stand or pulsation of an external force. If the characteristic region is entirely located within the stability zone, we can claim that with sufficiently small actuation amplitudes, the mechanism's oscillation amplitude will remain small. Here, the assumptions adopted as the initial ones in setting up the equations of motion (constancy of the inertial, quasielastic, and other coefficients of the mechanism) remain in force.

If a part of the characteristic region of the mechanism is found within the instability zone, then the methods of the linear theory cannot provide an answer to the question on the value of the amplitude of the established oscillations, since these methods do not take into account the effect of nonlinear factors on the motion of the mechanism. However, the fact that in the instability zone the oscillation amplitude may increase appreciably, is sufficient to strive to ensure its dynamic stability by the suitable selection of parameters in designing the mechanism. The necessity for this is enhanced also by the fact that at the boundaries of the instability zones, a resonance is possible, arising from the action of that actuation component, which depends only on time and is contained in the right member of equation (4.50).

§ 5.3. Resonance Condition.

The resonance condition for systems the behavior of which is described by the inhomogeneous Mathieu equation, according to equation (2.38) has the following appearance:

$$2ri + pi \pm \mu = 0.$$

It is not hard to notice that when $q = 0$, this condition changes over to the usual resonance condition for the system described by an inhomogeneous equation with constant coefficients. Actually, from equation (2.33) it follows that when $q = 0$

$$\operatorname{ch} \mu\pi = \cos \pi\sqrt{a},$$

from which /154

$$\mu = i\sqrt{a} = i\frac{2\omega_n}{\omega}.$$

Assuming in expression (2.38) that p = 2 and r = 0, we obtain the resonance condition

$$\omega = \omega_n.$$

Returning to equation (4.50) we see that frequency p of the actuating function in its right member coincides with the frequency of the periodic coefficient and is equal to 2. Consequently, the resonance condition assumes the appearance of

$$2(r + 1)i = \pm \mu$$

or, bearing in mind that r represents any whole number, we write

$$\mu = \pm 2ri.$$

Thus, resonance begins when the characteristic indicator is imaginary, whole, and even number; these values of the characteristic index correspond to the values of parameters a and q, with which there are periodic solutions of the homogeneous Mathieu equation (Mathieu functions with the π period).

In Figure 5.4 we give the stability chart with the resonance curves drawn with thick lines. Obviously, with the approach of the characteristic region of the mechanism to these lines, the oscillation amplitude with the same actuation intensity, will increase.

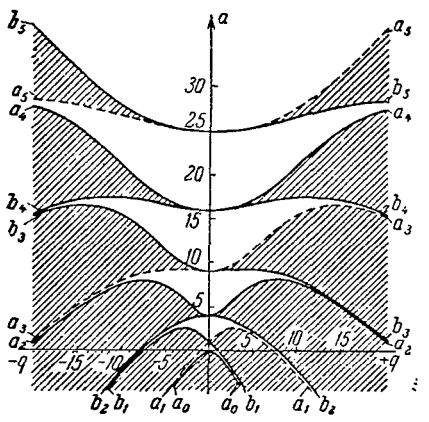

Figure 5.4.

It goes without saying that the linear theory in the analysis of the problem of dynamic stability of a mechanism and in the analysis of the conditions of resonance formation, yields only a qualitative evaluation of the mechanism's behavior, i.e., indicates only with which values of the mechanism's parameters and actuation parameters a substantial increase of oscillations is possible. Therefore, here we can note only that the characteristic region of the mechanism should be located as far as possible from the characteristic lines /155

a_0, a_2, b_2, ..., peculiar to periodic solutions.

§ 5.4. The Dynamic Equilibrium Position.

The problems connected with the dynamic stability and resonance of a mechanism are solved above without determination of the position of the dynamic equilibrium of a mechanism. However, the question itself of whether the position of dynamic equilibrium differs from the position of the statically stable equilibrium, and if it differs, then by how much, is extremely important. Indeed, in examining the problem of the stability of a mechanism it was assumed that the unknown position of the dynamic equilibrium is located within the bounds of the operating range of the mechanism.

However, it might occur, that with a certain actuation, the dynamic equilibrium position will be found outside the limits of the permissible range, which indicates the unsuitablility of the mechanism for operation under the operating conditions specified. We note that here the characteristic point of the mechanism may find itself in the instability zone. This will indicate that the corresponding position of dynamic equilibrium is unstable.

It is also important to determine the position of the dynamic equilibrium because its comparison with the position of static equilibrium, established in the case of the absence of secondary actuation, makes it possible to determine the magnitude of translation of the mechanism.

Finally, having determined a number of positions of dynamic equilibrium corresponding to various values of external constants or smoothly-varying forces applied to the mechanism's components, we can plot the dynamic characteristic of the mechanism and, having compared it with the statistical characteristic, get an idea of the dynamic precision of an instrument or device, which /156 includes the mechanism examined.

Let us first make only a formal determination of the dynamic equilibrium position, on the basis of the direct analysis of the equations of motion obtained in the preceding chapter. Somewhat later, using the simplest mechanical models, we shall attempt to find the causes, as the result of which, under the conditions of periodic actuation, the equilibrium position of the mechanism changes.

Finding the position of dynamic equilibrium involves the determination of a constant term of a particular solution of equation (4.50). For the convenience of the analysis we write this equation in the following form

$$\frac{d^2y}{dx^2} + (a - 2q_j \cos 2x)\, y = Q_0 + Q_j e^{2ix}. \qquad (5.1)$$

Since $e^{2ix} = \cos 2x + i \sin 2x$, the real part of the particular solution of equation (5.1) will represent the particular solution of equation (4.50) in

which we are interested. The particular solution of equation (5.1) by analogy with (2.37) may be written in the following way:

$$y_{\text{part}} = \frac{Q_0}{\delta}\left[\sum_{r=-\infty}^{\infty} c_{2r}\, e^{-2rix} \sum_{r=-\infty}^{\infty} \frac{c_{2r}}{2ri+\mu}\, e^{2rix} + \right.$$
$$\left. + \sum_{r=-\infty}^{\infty} c_{2r}\, e^{2rix} \sum_{r=-\infty}^{\infty} \frac{c_{2r}}{2ri+\mu}\, e^{-2rix}\right] +$$
$$+ \frac{Q_j}{\delta}\left[\sum_{r=-\infty}^{\infty} c_{2r}\, e^{-2rix} \sum_{r=-\infty}^{\infty} \frac{c_{2r}}{2(r+1)i+\mu}\, e^{2(r+1)ix} + \right.$$
$$\left. + \sum_{r=-\infty}^{\infty} c_{2r} e^{2rix} \sum_{r=-\infty}^{\infty} \frac{c_{2r}}{2(r-1)i+\mu}\, e^{-2(r-1)ix}\right]. \qquad (5.2)$$

The particular solution (5.2) contains four products of two periodic functions with a similar period. Since each of these products represents a periodic function with the same period, expression (5.2) may be written in the following abbreviated form: /157

$$y_{\text{part}} = \frac{Q_0}{\delta} \sum_{r=-\infty}^{\infty} H_{2r}^{Q_0} e^{2rix} + \frac{Q_j}{\delta} \sum_{r=-\infty}^{\infty} H_{2r}^{Q_j} e^{2(r+1)ix} = \sum_{r=-\infty}^{\infty} H_{2n} e^{2nix} \qquad (n = 0,\ 1,\ 2,\ \ldots).$$

Evidently, the particular solution of the inhomogeneous equation contains not one harmonic (as was the case in the particular solution of the inhomogeneous equation with constant coefficients), but an infinite series of harmonics, including the constant term H_0.

Let us determine this constant term, for which we shall separate the constant terms from four products, contained in brackets in equation (5.2). According to the adopted form of recording of the particular solution, H_0 represents the sum

$$H_0 = \frac{Q_0}{\delta} H_0^{Q_0} + \frac{Q_j}{\delta} H_0^{Q_j}.$$

The constant term in brackets with Q_0 is equal to

$$H_0^{Q_0} = 2 \sum_{r=-\infty}^{\infty} \frac{c_{2r}^2}{2ri+\mu}$$

and represents a sum of terms of an infinite series, containing products $e^{-2rix}\, e^{2rix} = 1$. The constant term in brackets with Q_j is equal to

$$H_0^{Qj} = \sum_{r=-\infty}^{\infty} c_{2r} \left[\frac{c_{2(r-1)}}{2(r-1)i+\mu} + \frac{c_{2(r+1)}}{2(r+1)i+\mu} \right].$$

Thus, the constant term of the particular solution is equal to

$$H_0 = \frac{2Q_0}{\delta} \sum_{r=-\infty}^{\infty} \frac{c_{2r}}{2ri+\mu} + \frac{Q_j}{\delta} \sum_{r=-\infty}^{\infty} c_{2r} \left[\frac{c_{2(r-1)}}{2(r-1)i+\mu} + \frac{c_{2(r+1)}}{2(r+1)i+\mu} \right].$$

In the case of a stable solution, the characteristic index μ represents an imaginary and nonintegral number ($\mu = i\beta$), and the free term H_0 proves equal to /158

$$H_0 = -\frac{2Q_0 i}{\delta} \sum_{r=-\infty}^{\infty} \frac{c_{2r}}{2r+\beta} - \frac{iQ_j}{\delta} \sum_{r=-\infty}^{\infty} c_{2r} \left[\frac{c_{2(r-1)}}{2(r-1)+\beta} + \frac{c_{2(r+1)}}{2(r+1)+\beta} \right]. \qquad (5.3)$$

Now let us turn to the presentation of two possible methods of determination of the position of dynamic equilibrium, for which we present a general solution of the inhomogeneous Mathieu equation in the form

$$y = f_1(x) + f_2(x).$$

Here

$$f_1(x) = Ae^{\mu x} \sum_{r=-\infty}^{\infty} c_{2r} e^{2rix} + Be^{-\mu x} \sum_{r=-\infty}^{\infty} c_{2r} e^{-2rix}$$

represents, according to (2.30), the general solution of the inhomogeneous equation, and

$$f_2(x) = H_0 + \sum_{r=-\infty}^{\infty *} H_{2r} e^{2rix}$$

represents a particular solution of an inhomogeneous equation (the asterisk signifies that the sum indicated contains no term corresponding to value r = 0).

The characteristic indicator μ and values c_{2r}, H_0, H_{2r} are certain functions of the value α_{dyn} sought, which characterizes the position of the dynamic equilibrium. We remind the reader that by α_{dyn} we understand the value

$$\alpha_{dyn} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \alpha \, dt.$$

According to this condition

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \alpha \, dt = 0,$$

or, in the designations adopted in the present chapter, /159

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x y\,dx=\lim_{x\to\infty}\frac{1}{x}\int_0^x f_1(x)\,dx+\lim_{x\to\infty}\frac{1}{x}\int_0^x f_2(x)\,dx=0.$$

Function $f_1(x)$ contains two arbitrary constants, and consequently, the latter equality is fulfilled in the case when each of the components, taken separately, is equal to zero. Bearing this in mind we get the following two equations:

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x f_1(x)\,dx=0,$$

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x f_2(x)\,dx=0.$$

As for the first equation, it is fulfilled only in the case that the characteristic index μ which corresponds to the value α_{dyn} found, proves an imaginary and nonintegral number. In the opposite case, the position of the mechanism, characterized by value α_{dyn}, proves unstable.

For the second equation, taking into account the periodic character of the particular solution, we obtain the equation for determining the position of the dynamic equilibrium, in the form of

$$H = 0.$$

In order to determine the position of the dynamic equilibrium, according to the method presented in [96], it is required to solve first the homogeneous equation in which parameters a and q represent certain functions of a constant but unknown function α_{dyn}. As the result of the solution, performed with an allowance for the finite number of terms, we can obtain an approximate expression for the characteristic index μ and coefficients c_{2r} in the form of certain functions α_{dyn}. Then, having substituted the function obtained in expression (5.3), we can obtain an equation for displacing H_0 in function α_{dyn}. In order to determine α_{dyn} after this, it is required to specify that $H = 0$, and solve the resulting transcendent equation. Having determined α_{dyn} we can calculate the numerical value of μ and find a full solution of the equation of motion.

The above-mentioned method of determination of α_{dyn}, as a rule, results /160 in boundless computations and may be used only in the simplest cases, as it has been done in work [96] applicably to the common pendulum.

Below we present another approximate method, based on the assumption that the position of dynamic equilibrium sought differs little from the static equilibrium position. Here, the difference giving us the mechanism's translation value sought

$$\Delta = \alpha_{dyn} - \alpha_0,$$

will represent the value of the same order of smallness, as the amplitude ε_0 of forced oscillations.

The assumption of the smallness of translation is quite acceptable, if only by the fact that the values of dynamic errors have extremely rigid standards. Consequently, if the actual translation value will be found within the tolerance limits, it must be a small value, and the method proposed enables us to appraise it. If, however, the value of the error is found not to be small, this will indicate the impermissible inaccuracy of the mechanism, and the task of determination of the precise value of α_{dyn} loses its practical significance.

In order to establish the essence of the proposed method of finding the dynamic error, we turn back to equation (5.1). Above it was indicated that in this equation, with unchanged actuation parameters all the coefficients are functions of the position of the mechanism, so that

$$a = a(\alpha_{dyn}) \quad q_j = q_j(\alpha_{dyn})$$
$$Q_0 = Q_0(\alpha_{dyn}) \quad Q_j = Q_j(\alpha_{dyn}).$$

Assuming that

$$\alpha_{dyn} = \alpha_0 + \Delta,$$

where Δ is a small value, we expand the functions indicated into a Taylor's series with respect to the small parameter Δ. We find

$$a = a(\alpha_0) + a'(\alpha_0)\Delta + \ldots = a_0 + a_0'\Delta + \ldots$$
$$q_j = q_j(\alpha_0) + q_j'(\alpha_0)\Delta + \ldots = q_{j0} + q_{j0}'\Delta + \ldots$$
$$Q_0 = Q_0(\alpha_0) + Q_0'(\alpha_0)\Delta + \ldots = Q_0 + Q_0'\Delta + \ldots$$
$$Q_j = Q_j(\alpha_0) + Q_j'(\alpha_0)\Delta + \ldots = Q_{j0} + Q_{j0}'\Delta + \ldots$$

We assume in the first approximation that the frequency of a mechanism's free oscillations, as well as the periodicity coefficient and actuation coefficient within the limits of the range of the mechanism's positions examined /161

$$\alpha_0 - \varepsilon_0 \leq \alpha \leq \alpha_0 + \Delta + \varepsilon_0$$

change little. Then we shall obtain the approximate equations

$$a \approx a_0, \quad \dot{q}_j \approx q_{j0}, \quad Q_j \approx Q_{j0}.$$

As for parameter

$$Q_0 = \frac{4d}{\omega^2} = \frac{4}{\omega^2} \frac{D}{J_{\text{dyn}}},$$

when $\alpha = \alpha_0$, $Q_0 = 0$; we can write the derivative of this parameter, bearing in mind that

$$\frac{dD}{d\alpha} = K,$$

may be written in the following way:

$$Q_0'(\alpha_0) = -\frac{4K}{J\omega^2},$$

where K and J are the quasielastic and inertial coefficients of the mechanism in the static equilibrium position. Taking into account the assumptions adopted, equation (4.50) will assume the appearance of

$$\frac{d^2y}{dx^2} + (a_0 - 2q_{j0}\cos 2x)\, y = Q_0'\Delta + Q_{j0}\cos 2x. \tag{5.4}$$

Now, the differential equation of motion of the mechanism contains as its parameters not the functions of the unknown value α_{dyn} but constant numbers, the values of which, dependent on the position of the mechanism's static equilibrium, were determined earlier.

Having solved the homogeneous equation (as before limiting ourselves to a finite number of terms), we find this time the numerical values of μ and coefficients c_{2r}. Having these values at our disposal, we can set up the expression, determining the value of the free term of the particular solution in the form of

$$H_0 = \frac{Q_0'\Delta}{\delta} H_0^{Q_0} + \frac{Q_{j0}}{\delta} H_0^{Q_{j0}}.$$

Having equated $H_0 = 0$ and substituted value Q_0' we find for the value of translation of the mechanism, the following expression /162

$$\Delta = \frac{H_0^{Q_{j0}}}{4H_0^{Q_0}} \frac{Q_{j0}}{K} J\omega^2. \tag{5.5}$$

Having substituted values Q_{j0} from expressions (4.48) and (4.49), we find that in the case of the pulsating load, the mechanism's translation is

$$\Delta = \frac{H_0^{Q_{10}}}{H_0^{\zeta_0}} \frac{P}{K} \left(\frac{ds_P}{d\alpha_0}\right)_0, \tag{5.6}$$

and in the case of the stand's vibration

$$\Delta = \frac{H_0^{Q_{20}}}{H_0^{Q_0}} \frac{\Theta_2}{K} \varrho_0 J \omega^2. \tag{5.7}$$

The method of linearization of the coefficients of the differential equation of motion used, enabled us to find the analytical expressions in the final form, determining the value of the mechanism's translation no matter how complicated the structure of the latter.

It is not hard to see, that in the case of a pulsating force, the translation value depends on the transmission ratio, connecting the position of the point of application of the force indicated and the conversion component; in the case of vibration of the stand, it depends on the value of the coefficients of periodicity and actuation, the expressions of which were obtained in the preceding chapter.

The values of these coefficients, just as the values of the transmission ratios, vary in relation to the position occupied by the mechanism in the process of its work, and in accordance with this the value of translation changes as well. Having calculated value Δ for various positions of the mechanism within the limits of its operating range, we obtain the characteristic of one of the two components of the complete dynamic error of a mechanism with elastic couplings.

We remark finally, that using formulas (5.6) and (5.7) and expressions determining Ψ_2 and Θ_2, we can formulate the problem of the synthesis of a mechanism, the dynamic errors of which, under vibration conditions, will be minimal. This problem is reduced to such a distribution of the masses of the mechanism's components, with which the periodicity and actuation coefficients, /163 representing trigonometric polynomials, deviate from zero the least.

However, the solution of the above problem goes outside the framework of that range of problems which is examined in the present book.

Let us turn back to formulas (5.6) and (5.7) obtained by us. In both cases the value of translation is proportional to the amplitude of actuation (P_0 or $\rho_0\omega^2$) and inversely proportional to the quasielastic coefficient of the mechanism. Moreover, the translation value depends on the constant parameters of the mechanism and its position.

Now, the position of the dynamic equilibrium of the mechanism in the first approximation may be determined from the equation

$$\alpha_{dyn} = \alpha_0 + \Delta_j \qquad (j = 1, 2).$$

When it is required to improve further the precision of the mechanism's translation value, all the above-mentioned calculations should be repeated, determining the values of parameters a, q_j, Q_0, and Q_j for $\alpha = \alpha_{dyn}$, etc.

The result obtained may be illustrated by the following geometric system, given in Figure 5.5. We shall assume at first, that the actuation is absent. Then the mechanism, being taken out of the position of stable equilibrium, performs harmonic oscillations about this position (see the lower curve in Figure 5.5). Equation (5.1) transforms into a free-oscillation equation and assumes the form of /164

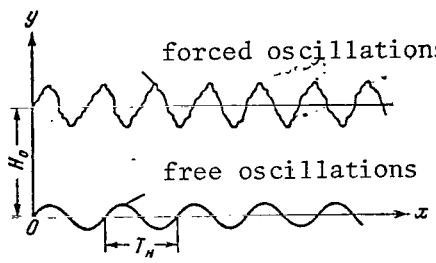

Figure 5.5.

$$\frac{d^2y}{dx^2} + ay = 0,$$

and, as indicated earlier, parameter a depends on the value of the generalized coordinate α_0, which characterizes the position of the static equilibrium of the mechanism.

In the presence of harmonic actuation the picture undergoes substantial changes:

first, the forced oscillations are performed about the position of the dynamic equilibrium of the mechanism (upper curve in Figure 5.5), which may differ from the static equilibrium position;

second, forced oscillations, the amplitude of which, in the case of stability of the mechanism remains limited when $t \to \infty$, may be not only a non-harmonic, but also not a periodic time function.

§ 5.5. The Amplitude of Forced Oscillations.

Now, we shall begin the determination of the value of disalignment of the mechanism. For this purpose we shall write the particular solution of equation (5.4) assuming that the dynamic stability of the mechanism is assured and that the forced oscillations of the mechanism have attained a settled character.

Earlier it was indicated that the particular solution, in addition to the constant member H_0, contains a series of harmonics, the frequencies of which

are multiples of the actuation frequency, which has a harmonic character. The problem of determination of the amplitude of forced oscillations is reduced to the determination of coefficients H_{2n} of the particular solution

$$y_{\text{part}} = \sum_{n=-\infty}^{\infty} H_{2n} e^{2nix}.$$

In order to get rid of the necessity of separating the real and imaginary parts of the solution, we present the right member of equation (5.4) in the following form:

$$Q_0'\Delta + Q_{j0}\cos 2x = Q_0'\Delta + Q_{j0}\frac{e^{2ix}+e^{-2ix}}{2}.$$

In this case, for the particular solution we obtain the following expression: /165

$$\begin{aligned} y_{\text{part}} = &-\frac{iQ_0'\Delta}{\delta}\Big[\sum_{r=-\infty}^{\infty} c_{2r}e^{-2rix}\sum_{r=-\infty}^{\infty}\frac{c_{2r}}{2r+\beta}e^{2rix} + \\ &+\sum_{r=-\infty}^{\infty} c_{2r}e^{2rix}\sum_{r=-\infty}^{\infty}\frac{c_{2r}}{2r+\beta}e^{-2rix}\Big] - \\ &-\frac{iQ_{j0}}{2\delta}\Big[\sum_{r=-\infty}^{\infty} c_{2r}e^{-2rix}\sum_{r=-\infty}^{\infty}\frac{c_{2r}}{2(r+1)+\beta}e^{2(r+1)ix} + \\ &+\sum_{r=-\infty}^{\infty} c_{2r}e^{2rix}\sum_{r=-\infty}^{\infty}\frac{c_{2r}}{2(r-1)+\beta}e^{-2(r-1)ix} + \\ &+\sum_{r=-\infty}^{\infty} c_{2r}e^{-2rix}\sum_{r=-\infty}^{\infty}\frac{c_{2r}}{2(r-1)+\beta}e^{2(r-1)ix} + \\ &+\sum_{r=-\infty}^{\infty} c_{2r}e^{2rix}\sum_{r=-\infty}^{\infty}\frac{c_{2r}}{2(r+1)+\beta}e^{-2(r-1)ix}\Big]. \end{aligned} \tag{5.8}$$

To determine coefficient H_{2n} it suffices to separate in this particular solution the corresponding terms, and arrange them in groups. Let us determine, for example H_{20}, for which we separate from expression (5.8) the coefficients of function e^{2ix}. As the result we get

$$H_{20} = -\frac{iQ_0'\Delta}{\delta}\sum_{r=-\infty}^{\infty} c_{2r}\frac{c_{2(r-1)}+c_{2(r+1)}}{2r+\beta} - \frac{iQ_{j0}}{2\delta}\sum_{r=-\infty}^{\infty} c_{2r}\left[\frac{c_{2(r-2)}+c_{2r}}{2(r+1)+\beta} + \frac{c_{2r}+c_{2(r+2)}}{2(r-1)+\beta}\right].$$

Having separated from (5.8) the coefficients of function e^{-2ix}, we find

$$H_{-20}=-\frac{iQ_0'\Delta}{\delta}\sum_{r=-\infty}^{\infty}c_{2r}\frac{c_{2(r-1)}+c_{2(r+1)}}{2r+\beta}-\frac{iQ_{j_0}}{2\delta}\sum_{r=-\infty}^{\infty}c_{2r}\left[\frac{c_{2(r-2)}+c_{2r}}{2(r+1)+\beta}+\frac{c_{2r}+c_{2(r+2)}}{2(r-1)+\beta}\right].$$

It is not hard to see that $H_{-20} = H_{20}$. In a similar manner we can calculate coefficients H_{40}, H_{-40}, H_{60}, H_{-60}, etc. /166

The expression of any coefficient H_{2n} of the sum representing the particular solution of equation (5.4) has the following appearance:

$$H_{2n}=-\frac{iQ_0'\Delta}{\delta}\sum_{r=-\infty}^{\infty}c_{2r}\frac{c_{2(r-n)}+c_{2(r+n)}}{2r+\beta}-\frac{iQ_{j_0}}{2\delta}\sum_{r=-\infty}^{\infty}c_{2r}\left[\frac{c_{2(r-1-n)}+c_{2(r-1+n)}}{2(r-1)+\beta}+\frac{c_{2(r+1+n)}+c_{2(r+1-n)}}{2(r+1)+\beta}\right]. \quad (5.9)$$

Specifying that in equation (5.9) n = 0, we find the expression for determining the constant term of the particular solution, coincident with expression (5.3). Bearing in mind that

$$\Delta=-\frac{Q_{j_0}}{Q_0'}\frac{H_0^{Q_{j_0}}}{H_0^{Q_0}},$$

and using equation (5.9), we can present solution (5.8) in the following form:

$$y_{\text{part}}=\frac{iQ_{j_0}}{2\delta}\sum_{n=-\infty}^{\infty}{}^{*}\left\{\frac{2H_0^{Q_{j_0}}}{H_0^{Q_0}}\sum_{r=-\infty}^{\infty}\frac{c_{2(r+n)}+c_{2(r-n)}}{2r+\beta}-\right.$$
$$\left.-\sum_{r=-\infty}^{\infty}c_{2r}\left[\frac{c_{2(r-1-n)}+c_{2(r-1+n)}}{2(r-1)+\beta}+\frac{c_{2(r+1+n)}+c_{2(r+1-n)}}{2(r+1)+\beta}\right]\right\}e^{2nix}. \quad (5.10)$$

The asterisk signifies that the given sum does not contain term n = 0.

In solution (5.10) value δ represents the determinant of the fundamental system y_1 and y_2 of a homogeneous equation, corresponding to equation (5.4). This fundamental system, as it was shown in Chapter 2, with $\mu = i\beta$ will assume the following form

$$y_1=\sum_{r=-\infty}^{\infty}c_{2r}e^{(2r+\beta)ix},\qquad y_2=\sum_{r=-\infty}^{\infty}c_{2r}e^{-(2r+\beta)ix}. \quad (5.11)$$

By differentiating we obtain /167

$$\left.\begin{aligned}\frac{dy_1}{dx} &= i\sum_{r=-\infty}^{\infty}(2r+\beta)c_{2r}e^{(2r+\beta)ix},\\ \frac{dy_2}{dx} &= -i\sum_{r=-\infty}^{\infty}(2r+\beta)c_{2r}e^{-(2r+\beta)ix}.\end{aligned}\right\} \tag{5.12}$$

The determinant of the fundamental system, according to the determination is equal to

$$\delta = y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx} \tag{5.13}$$

and remains constant with any values of independent variable x.

Substituting (5.11) and (5.12) in (5.13) we obtain, with x = 0, the following value of δ:

$$\delta = -2i\sum_{r=-\infty}^{\infty}c_{2r}\sum_{r=-\infty}^{\infty}c_{2r}(2r+\beta). \tag{5.14}$$

Rewriting (5.10) in the trigonometric form and substituting the value of δ obtained, we present the equation of the forced oscillations of the mechanism in the following form:

$$y_{\text{part}} = -\frac{Q_{j0}}{2\sum_{r=-\infty}^{\infty}c_{2r}\sum_{r=-\infty}^{\infty}(2r+\beta)c_{2r}}\times$$
$$\times\sum_{n=1}^{\infty}{}^{*}\left\{\frac{2H_0^{Q_{j0}}}{H_0^{Q_0}}\sum c_{2r}\frac{c_{2(r+n)}+c_{2(r-n)}}{2r+\beta}-\right.$$
$$\left.-\sum_{r=-\infty}^{\infty}c_{2r}\left[\frac{c_{2(r-1-n)}+c_{2(r-1+n)}}{2(r-1)+\beta}+\frac{c_{2(r+1+n)}+c_{2(r+1-n)}}{2(r+1)+\beta}\right]\right\}\cos 2nx. \tag{5.15}$$

Specifying in this equation that x = 0, we find the amplitude of the mechanism's forced oscillations. Doubling this amplitude, we obtain the value of the mechanism's disalignment sought, which we designate with E. It will be equal to: /168

$$E = -\frac{Q_{j0}}{\sum_{r=-\infty}^{\infty}c_{2r}\sum_{r=-\infty}^{\infty}c_{2r}(2r+\beta)}\times\sum_{n=1}^{\infty}{}^{*}\left\{\frac{2H_0^{Q_{j0}}}{H_0^{Q_0}}\sum_{r=-\infty}^{\infty}c_{2r}\frac{c_{2(r+n)}+c_{2(r-n)}}{2r+\beta}-\right.$$
$$\left.-\sum_{r=-\infty}^{\infty}c_{2r}\left[\frac{c_{2(r-1-n)}+c_{2(r-1+n)}}{2(r-1)+\beta}+\frac{c_{2(r+1+n)}+c_{2(r+1-n)}}{2(r+1)+\beta}\right]\right\}. \tag{5.16}$$

In practical calculations the disalignment value may be determined only approximately, limiting ourselves in the determination of the amplitude of one harmonic or another, to the estimation of several central coefficients c_{2r} and adding up these amplitudes for a limited number of harmonics.

In conclusion we indicate that in examining the problems of translation and disalignment of a mechanism we have actually determined the translation and disalignment of only one of its mobile components, the position of which is characterized by the generalized coordinate α. In a number of cases it might become necessary to determine the translation and disalignment of some other component included in the kinematic chain of the mechanism.

The relationship between Δ and E of a transmission component and the respective values Δ_i and E_i of the i-th component of the mechanism is established by the following relationships. In determining the translation of the i-th component

$$\Delta_i = \Delta \left(\frac{d\alpha_i}{d\alpha} \right)_0$$

for the case when the i-th component has a rotary motion, or

$$\Delta_i = \Delta \left(\frac{ds_i}{d\alpha} \right)_0 ,$$

when the i-th component moves forward. Correspondingly, in the determination of the disalignment of the i-th component

$$E_i = E \left(\frac{d\alpha_i}{d\alpha} \right)_0 ,$$

or

$$E_i = E \left(\frac{ds_i}{d\alpha} \right)_0 .$$

Thus, in this and the preceding paragraph, we have solved the problem of determination of two components of the full dynamic error. The relationships obtained enable us to plot by the conventional methods the corresponding frequency characteristics of a mechanism with elastic couplings. /169

§ 5.6. On the Causes of Formation of Translation.

Up to now the problems of determining translation and disalignment of a mechanism were solved by us by the formal method, proceeding from the conventional method of analysis of the differential equation of motion. As applied to translation, the investigation performed above does not give a sufficiently illustrative idea on the causes producing this phenomenon.

However, if in addition to the above-adopted assumption on the smallness of the amplitude of the forced oscillations of the mechanism, we introduce certain additional assumptions, the analysis of the translation phenomenon and its causes in some cases proves extremely simple and illustrative. One of such instances is examined below, the assumptions used coinciding completely with or being very close to those which are adopted in works [34, 94].

In distinction from the first one of the works mentioned, we do not resort here to the approximate integration of the equation of motion, but limit ourselves to the examination of the approximate kinematic system of the process of motion and the phenomena accompanying it.

The existence of an elastic coupling distinguishes our model substantially from the model examined in both works.

Let us reproduce again the system of the mechanism presented in Figure 4.8, while in this case we may exclude from examination the two-driver group, the mass of which is small in comparison with the mass concentrated in point A (Figure 5.6). Here, the system will represent a mathematical pendulum with an elastic coupling and a vibrating suspension.

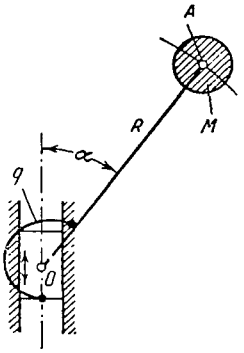

Figure 5.6.

Let us assume that the frequency of the free oscillations of this pendulum is small in comparison with the frequency of oscillations of the stand (i.e., $\omega_n \ll \omega$). Then the forced motion of the pendulum will be displaced in phase by 180° with respect to the motion of the suspension point. /170

Let us assume also that the actuation amplitude is small in comparison with the length of the pendulum (i.e., $\rho_0 \ll R$).

In Figure 5.7, in accordance with the assumptions indicated we give the "middle" and two extreme positions of the pendulum which it occupies in the process of motion.

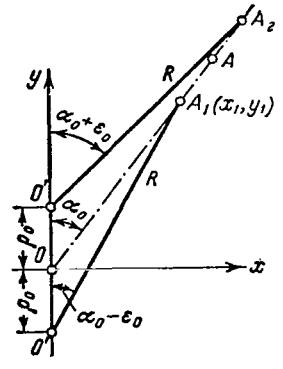

Figure 5.7.

At first we establish, what will take place if by some way (for example, by applying additional rigid couplings) we ensure, in the process of vibration, the constancy of the middle position of the pendulum, corresponding with the position of static equilibrium. We select an immobile system of coordinate, the source of which coincides with the middle position of the point of suspension of the pendulum, and set up an equation of the straight line passing through points A_1 and A_2, and which mass M in the extreme positions will coincide. The coordinates of these points are equal to

$$x_1 = R\sin(\alpha_0 - \varepsilon_0), \qquad x_2 = R\sin(\alpha_0 + \varepsilon_0),$$
$$y_1 = -\varrho_0 + R\cos(\alpha_0 - \varepsilon_0), \qquad y_2 = \varrho_0 + R\cos(\alpha_0 + \varepsilon_0),$$

where amplitude ε_0 is the amplitude of the forced oscillations of the pendulum.

Specifying that $\cos \varepsilon_0 = 1$, $\sin \varepsilon_0 = \varepsilon_0$ we obtain the following expressions for determining the coordinates of points A_1 and A_2:

$$x_1 = R\sin\alpha_0 - R\varepsilon_0\cos\alpha_0; \qquad x_2 = R\sin\alpha_0 + R\varepsilon_0\cos\alpha_0,$$
$$y_1 = -\varrho_0 + R\cos\alpha_0 + R\varepsilon_0\sin\alpha_0; \quad y_2 = \varrho_0 + R\cos\alpha_0 - R\varepsilon_0\sin\alpha_0.$$

The equation of the straight line passing through the two prescribed points has the following appearance

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

Substituting here the values of the coordinates and carrying out the transformations we obtain the following equation of the straight line /171

$$y = (\cot \alpha_0)\, x.$$

Consequently, the straight line connecting the two extreme positions of mass M passes through the source of the coordinates and, moreover, coincides with the middle position of the pendulum.

Bearing in mind that the three positions (two extreme and one middle) of point A are located on a common straight line, and taking into account the relative smallness of the complete displacement of point A, we make yet another assumption, according to which the trajectory of mass M in the course of the entire period of motion, corresponds with line A_1A_2.

With this assumption, the kinematics of a system is visually illustrated by means of a four-component mechanism, shown in Figure 5.8. In this mechanism, slider 1 represents a vibrating stand, connecting rod 2, the rod of the pendulum, slider 3, the mass M moving along the A_1A_2 line.

If we designate with S the distance of slider 3 from its middle position, then according to Figure 5.8,

$$S = OA - R,$$
$$OA = \varrho\cos\alpha_0 + R\cos\varepsilon.$$

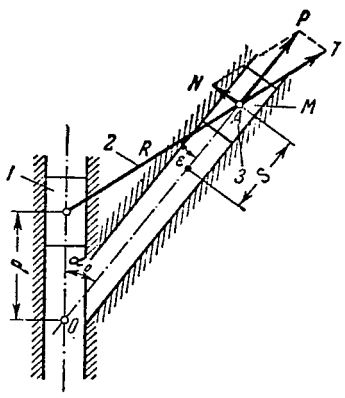

Figure 5.8.

When $\cos\varepsilon \approx 1$

$$OA \approx \varrho\cos\alpha_0 + R,$$
$$S = \varrho\cos\alpha_0, \quad \dot{S} = \dot{\varrho}\cos\alpha_0, \quad \ddot{S} = \ddot{\varrho}\cos\alpha_0.$$

The current value of the angle of turn of the connecting rod will be obtained from the geometric relationships

$$\varepsilon \approx \sin\varepsilon = \frac{\varrho}{R}\sin\alpha_0.$$

Let us find the force of pressure from the direction of slider 3 on the guide-rail, considering that mass M is concentrated in the center of the slider. For this we apply at point A of the slider a force P equal to /172

$$P = -M\ddot{S},$$

and find its components T and N, directed along the connecting rod and perpendicularly to the guide rail respectively. These components are equal to

$$T = \frac{M\ddot{S}}{\cos\varepsilon} \approx M\ddot{S} = M\ddot{\varrho}\cos\alpha_0,$$
$$N = M\ddot{S}\tan\varepsilon \approx M\ddot{S}\varepsilon = M\ddot{\varrho}\frac{\varrho}{R}\sin\alpha_0\cos\alpha_0.$$

Let us examine the case of the harmonic law of motion of slider 1:

$$\rho = \rho_0 \cos \omega t.$$

Here, the component, acting along the connecting rod is equal to

$$T = M\varrho_0\omega^2\cos\alpha_0\cos\omega t.$$

The mean value of this component for the entire period τ_0 of the motion of slider 1 is equal to

$$\frac{M\varrho_0\omega^2\cos\alpha_0}{\tau_0}\int_0^{\tau_0}\cos\omega t\,dt = 0.$$

Component N, acting on the guide rail, is equal to

$$N = M\frac{\varrho_0^2}{R}\omega^2\sin\alpha_0\cos\alpha_0\cos^2\omega t,$$

its mean value N_{dyn} is equal to

$$N_{dyn} = M \frac{\varrho_0^2}{R\tau_0} \omega^2 \sin \alpha_0 \cos \alpha_0 \int_0^{\tau_0} \cos^2 \omega t \, dt,$$

from which

$$N_{dyn} = \frac{M}{4R} \varrho_0^2 \omega^2 \sin 2\alpha_0.$$

The moment within the pivot O of the force N_{dyn}, which we shall call the dynamic moment of the mechanism, is equal to /173

$$M_{dyn} = -\frac{M}{4} \varrho_0^2 \omega^2 \sin 2\alpha_0.$$

The value of the dynamic moment*, arising in the process of the mechanism's moment examined, depends on the position and amplitude of actuation, while its action on the mechanism is analogous to the action of the moment of the usual forces balanced in our example, by the reaction of the guide rail of slider 3.

Knowing the value of the dynamic moment we can return to the examination of the pendulum with an elastic coupling, given in Figure 5.6.

With an immobile suspension the position of static equilibrium α_0 is determined from the obvious equality

$$GR \sin \alpha_0 - q\alpha_0 = 0.$$

When the suspension of the pendulum begins to vibrate, the dynamic moment arises here, and acting in the given case not on the absolutely rigid guide rail but on an elastic system, will tend to take the pendulum out of the position of static equilibrium α_0 and set it in the direction of the axis, along which the suspension point vibrates.

As the result of this action a certain new dynamic position of equilibrium will be established, coinciding with the new "middle" position of equilibrium. This position may be determined having solved the equation of dynamic equilibrium, which has the following form:

$$GR \sin \alpha_{dyn} - q\alpha_{dyn} - \frac{M}{4} \varrho_0^2 \omega^2 \sin 2\alpha_{dyn} = 0. \qquad (5.17)$$

---

* The expression of the dynamic moment obtained above coincides completely with the expression given in work [34] and is called the vibration moment in it.

This simple example may serve as an illustrative proof that the cause of translation of the mechanism with elastic couplings, operating under the conditions of the stand's vibration, is the action of the dynamic moment absorbed by the elastic couplings of the mechanisms, which are deformed under its effect. /174

Let us assume that the position of dynamic equilibrium in the example examined differs little from the position of stable equipment of the mechanism. Then if we assume, as before, that

$$\alpha_{dyn} = \alpha_0 + \Delta,$$

where Δ is a small value, then, having made a suitable substitution in equation (5.17) and assuming that $\cos \Delta \approx 1$, and $\sin \Delta \approx \Delta$, we get

$$\left(GR\cos\alpha_0 - q - \frac{M}{2}\varrho_0^2\omega^2\cos 2\alpha_0\right)\Delta - \frac{M}{4}\varrho_0^2\omega^2\sin 2\alpha_0 = 0, \qquad (5.18)$$

from which

$$\Delta = \frac{\frac{M}{4}\varrho_0^2\omega^2\sin 2\alpha_0}{GR\cos\alpha_0 - q - \frac{M}{2}\varrho_0^2\omega^2\cos 2\alpha_0}. \qquad (5.19)$$

As an example, let us prescribe the following dimensions for values: R = 5.0 cm, G - 50 grams, $\alpha_0 = 40°$, $\rho_0 = 0.1$ cm. Then the required rigidity of the elastic coupling will be equal to

$$q = \frac{GR\sin\alpha_0}{\alpha_0} = 230{,}3\ \text{g}\cdot\text{cm/rad}.$$

In Figure 5.9, the curve of translation of the pendulum, calculated according to equation (5.19) is drawn with a broken line. The solid line gives the values of translations, obtained as the result of approximate integration of the equation of motion (see below).

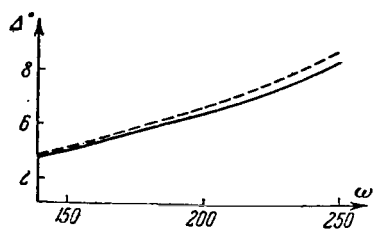

Figure 5.9.

From this example it is clear that even with a relatively large disbalance of the system (R = 5.0 cm) and weight, and also with a relatively small rigidity of the elastic coupling, the translation value attains an appreciable size only with a substantial frequency of actuation. Thus, applicably to the mechanisms with sufficiently rigid elastic couplings, the disbalance of the system cannot serve as the cause of origination of any substantial translation. /175

In conclusion, let us examine the case of the absence of an elastic coupling ($q = 0$). With an immobile suspension, the pendulum, in the case of the absence of an elastic coupling, has two positions of equilibrium: $\alpha_0 = 0$, and $\alpha_0 = 180°$. The first one of them is the position of unstable equilibrium.

Equation (5.18) enables us to establish the condition with which the position of the statically unstable equilibrium of a turned pendulum, becomes stable as the result of vibration of the suspension point.

To determine this condition it suffices to specify that $\alpha_0 = 0$ and $q = 0$. Then equation (5.18) with $\Delta \neq 0$, is fulfilled, if

$$GR - \frac{M}{2} \varrho_0^2 \omega^2 = 0,$$

from which we obtain the condition of dynamic "stabilization" of the pendulum's equilibrium, when its center of gravity is located above the fulcrum

$$\varrho_0^2 \omega^2 \gg 2gR.$$

§ 5.7. The Case of $\alpha_0 << 1$, $q_{20} << 1$.

The investigation of the dynamic precision of a mechanism with elastic couplings operating under the conditions of vibration of the stand or pulsation of the external force, cannot be called completed until the results obtained by two different ways have been combined into a single entity. It is required to coordinate two approximate solutions, one of which is built on the formal application of the method developed for solving the problem of determining the dynamic errors (§ 5.4) and the other is found by means of a simple mechanical model, examined in the preceding paragraph.

If we succeed in coordinating the solutions mentioned, we shall obtain, in this way, the confirmation of the fact that the analysis of the causes of the translation phenomenon, as well as the procedure of the approximate determination of the values of translation and disalignment are built on assumptions reflecting correctly the essence of the phenomenon.

We shall perform the coordination of the two solutions using the example of the mathematical pendulum with an elastic coupling and a vibrating suspension, which were examined in detail in the preceding paragraph. We /176 shall remind the reader about the initial assumptions:

1) the frequency of the free oscillations of a pendulum with an elastic coupling is small in comparison with the frequency of the vibration of the suspension point; 2) the length of the pendulum is great in comparison with the amplitude of vibration of the suspension point; 3) the resulting motion of

mass M proceeds along a straight line coincident with the middle position of the pendulum.

In Figure 5.6 we have determined the dynamic errors of the pendulum with an elastic coupling, making use of the illustrative geometric and kineto-static considerations. Now, in order to determine the dynamic errors of this mechanism, we shall use the previously-developed method of dynamic analysis, beginning with the determination of the translation value. According to expression (5.19) the value of translation depends on the intensity of the vibration, the rigidity of the elastic bonds, the value of the mobile mass, and on the middle position, occupied by the pendulum in the process of vibration of the suspension point.

Earlier (see § 5.4) it was established that if the expected translation value is not great, it may be determined using expression (5.7).

For the case of the vibration of the stand this expression assumes the form of

$$\Delta = \frac{H_0^{Q_{20}}}{H_0^{Q_0}} \frac{\Theta}{K} \varrho_0 \omega^2 J. \tag{5.20}$$

Here, the following values are unknown

$$H_0^{Q_0} = 2\sum_{r=-\infty}^{\infty} \frac{c_{2r}^2}{2ri+\mu},$$

$$H_0^{Q_{20}} = \sum_{r=-\infty}^{\infty} c_{2r}\left[\frac{c_{2(r-1)}}{2(r-1)i+\mu} + \frac{c_{2(r+1)}}{2(r+1)i+\mu}\right],$$

the calculation of which is the principal task of the present paragraph. To solve this problem it is required to determine the value of the characteristic index μ and coefficients c_{2r}, i.e., to find the solution of the homogeneous equation

$$\frac{d^2y}{dx^2} + (a_0 - 2q_{20}\cos 2x)\, y = 0, \tag{5.21}$$

basing our solution on the assumptions adopted as the intial one in examination/177 of the model in Figure 5.8. It is not hard to notice that the above-mentioned assumptions on the relationships between the parameters of the mechanism and parameters of actuation are reflected quite definitively in the values of parameters a_0 and q_{20} of equation (5.21).

Indeed, the assumption that frequency ω_{n0} of the free oscillations of the pendulum is substantially smaller than the frequency of vibrations of its suspension point, results in the fact that the value of parameter a_0 proves small, i.e.,

$$a_0 = \left(\frac{2\omega_{n0}}{\omega}\right)^2 \ll 1.$$

The assumption that amplitude ρ_0 of vibration of the suspension point is small in comparison with length R of the pendulum, results in the fact that the value of parameter q_{20} with respect to absolute value also proves small

$$|q_{20}| = 2\varrho_0, \quad |\Psi_{20}| = 2\frac{\varrho_0}{R}, \quad |\cos\alpha_0| \ll 1.$$

Values ω_{n0} and Ψ_{20} included in the expressions of parameters a_0 and q_{20}, for the case of the pendulum with elastic couplings and a vibrating suspension point, were determined above, in the example given in § 4.5. Thus, we should find the solution of equation (5.21) for the case when both parameters of this equation prove to be small values in comparison with a unity.

For the solution let us use the method presented in the second chapter, and with the purpose of giving a visual illustration of the course of the solution, let us adduce certain intermediate computations.

As we know (see (2.23)), if $q_{20} << 1$, the relationship between parameters a_0 and q_{20} of the mechanism is expressed by the ratio

$$\cos(i\mu\pi) \approx \cos\pi\sqrt{a_0} + \frac{\pi q_{20}^2 \sin(\pi\sqrt{a_0})}{4\sqrt{a_0}(a_0-1)}.$$

If $a_0 << 1$, then, specifying that $\sin\pi\sqrt{a_0} \approx \pi\sqrt{a_0}$, we can write

$$\cos(i\mu\pi) \approx \cos\pi\sqrt{a_0} - \frac{\pi^2}{4}q_{20}^2.$$

Disregarding the second term containing the square of the small value, we get /178

$$i\mu \approx \sqrt{a_0}.$$

Consequently, with the adopted values of the parameters the characteristic index proves to be an imaginary value

$$\mu = i\beta,$$

where $\beta = -\sqrt{a_0}$.

To determine coefficients c_{2r} we use equation (2.31), having substituted $i\mu = \sqrt{a_0}$ into which we get

$$c_{2r}\left[a_0-(2r-\sqrt{a_0})^2\right]-q_{20}\left[c_{2(r-1)}+c_{2(r+1)}\right]=0,$$

or

$$4rc_{2r}(\sqrt{a_0}-r)=q_{20}\left[c_{2(r-1)}+c_{2(r+1)}\right]. \tag{5.22}$$

In the second chapter it was shown that, if parameters a_0 and q_{20} are not very big, with the increase of r, coefficients c_{2r} decrease rather rapidly. In the case when $q_{20} << 1$, coefficients c_{2r}, with the increase of r, decrease even more rapidly, and therefore, it is quite possible, taking into consideration the relatively small number of terms, to obtain a sufficiently accurate result. In order to form an idea of how rapidly coefficients c_{2r} decrease, we calculate them for values of r equal to -2, -1, 0, 1, 2.

Substituting in equation (5.22) the value r = -2, and ignoring term c_{-6}, we get

$$-8c_{-4}(\sqrt{a_0}+2)=q_{20}c_{-2},$$

from which

$$c_{-4}=-\frac{q_{20}}{8(\sqrt{a_0}+2)}c_{-2}.$$

Substituting value r = -1 into equation (5.22) we get

$$-4c_{-2}(\sqrt{a_0}+1)=q_{20}(c_{-4}+c_0),$$

from which

$$c_{-2}=\frac{c_0+c_4}{4(\sqrt{a_0}+1)}q_{20}.$$

Substituting values c_{-4} we get /179

$$\left[1+\frac{q_{20}^2}{32(\sqrt{a_0}+1)(\sqrt{a_0}+2)}\right]c_{-2}=-\frac{q_{20}}{4(\sqrt{a_0}+1)}c_0.$$

Disregarding the term containing the square of value $q_{20} << 1$ we find

$$c_{-2}=-\frac{q_{20}}{4(\sqrt{a_0}+1)}c_0.$$

The next coefficient c_{-4}, expressed with coefficient c_0 is found equal to

$$c_{-4}=\frac{q_{20}^2}{32(\sqrt{a_0}+1)(\sqrt{a_0}+2)}c_0,$$

and in the first approximation we may disregard it.

In a similar way we calculate coefficients c_4 and c_2. Substituting value r = 2 into equation (5.22) and disregarding the value of coefficient c_6, we get

$$c_4=\frac{q_{20}}{8(\sqrt{a_0}-2)}c_2.$$

Substituting r = 1 into the same equation, we get

$$c_2=\frac{c_0-c_4}{4(\sqrt{a_0}-1)}q_{20}.$$

Disregarding the term containing coefficient

$$c_4=\frac{q_{20}^2}{32(\sqrt{a_0}-1)(\sqrt{a_0}-2)}c_0,$$

we get

$$c_2=\frac{q_{20}}{4(\sqrt{a_0}-1)}c_0.$$

It is not hard to see that with such an approximate determination coefficients c_2 and c_{-2}, c_4 and c_{-4}, etc., differ somewhat in value from one another, but bearing in mind condition $a_0 \ll 1$, we may expect the error not to be large in this case.

Having determined the characteristic index μ and coefficients c_{2r}, and consequently, having obtained the solution of equation (5.21), we can turn to the calculation of coefficients $H_0^{Q}20$ and $H_0^{QQ}0$. Here, we shall limit ourselves to the estimation of only three central coefficients /180

$$c_{-2}=-\frac{q_{20}}{4(\sqrt{a_0}+1)}c_0,\quad c_0,\quad c_2=\frac{q_{20}}{4(\sqrt{a_0}-1)}c_0,$$

disregarding the terms containing the squares and higher powers of the small value q_{20}. Bearing in mind that $\beta = -\sqrt{a_0}$, we get for coefficient $H_0^{QQ}0$ the expression

$$H_0^{Q_0}=-2i\sum_{r=-1}^{1}\frac{c_{2r}^2}{2r-\sqrt{a_0}}=2i\left[\frac{c_{-2}^2}{\sqrt{a_0}+2}+\frac{c_0^2}{\sqrt{a_0}}+\frac{c_2^2}{\sqrt{a_0}-2}\right].$$

Disregarding the terms containing the squares of coefficients c_{-2} and c_2, proportional to value $q_{20} << 1$, we get

$$H_0^{Q_0} = 2i \frac{c_0^2}{\sqrt{a_0}}.$$

For calculating coefficient $H_0^{Q_{20}}$, disregarding terms containing c_{-4} and c_4, we obtain the following expression:

$$H_0^{Q_{20}} = i \sum_{r=-1}^{1} c_{2r} \left[\frac{c_{2(r-1)}}{2(r-1)-\sqrt{a_0}} + \frac{c_{2(r+1)}}{2(r+1)-\sqrt{a_0}} \right] = 2ic_0 \left[c_{-2} \frac{\sqrt{a_0}+1}{\sqrt{a_0}(\sqrt{a_0}+2)} + c_2 \frac{\sqrt{a_0}-1}{\sqrt{a_0}(\sqrt{a_0}-2)} \right].$$

Substituting values c_{-2} and c_2, we get

$$H_0^{Q_{20}} = \frac{2q_{20}}{\sqrt{a_0}(a_0-4)} ic_0^2.$$

Bearing in mind that $a_0 << 1$

$$H_0^{Q_{20}} = -\frac{q_{20}}{2\sqrt{a_0}} ic_0^2.$$

Consequently, for the case examined ($a_0 << 1$, $q_{20} << 1$) the relationship between the coefficients is

$$\frac{H_0^{Q_{20}}}{H_0^{Q_0}} = -\frac{q_{20}}{4} = \frac{\varrho_0}{2R} \cos \alpha_0.$$

Substituting this value in expression (5.20), we get

$$\Delta_2 = \frac{\Theta_2}{2K} \frac{\varrho_0^2 \omega^2 J}{R} \cos \alpha_0.$$

The value of the coefficient of actuation Θ_2 for the instance examined, /181 under the condition that $\omega_n < \omega$, was determined by us in § 4.5 and was found equal to

$$\Theta_2 = -\frac{\sin \alpha_0}{R}.$$

The inertial and quasielastic coefficients were respectively equal to

$$J = MR^2, \qquad K = q - GR \cos \alpha_0.$$

After the substitution of these values the expression for the translation assumes the following appearance:

$$\Delta_2 = \frac{\frac{M}{4}\varrho_0^2\omega^2 \sin 2\alpha_0}{GR\cos\alpha_0 - q}. \qquad (5.23)$$

Comparing expression (5.23) and (5.19) we can establish the almost complete identity of both equations. With the value $\alpha_0 = \pi/4$ they are found to be identical (in this position of the pendulum the translation, the other conditions being equal, reaches the maximum value). With values $\alpha_0 \leq \pi/4$, the values of translations, calculated according to equations (5.23) and (5.19), may differ substantially one from another. This difference may be explained by the fact that in Figure 5.8 slider M is forced to move along a straight line, whereas in reality, even when conditions $\omega_{n0} << \omega$, $\rho_0 << R$ are fulfilled, mass M of the pendulum will move not along the straight line but over a certain trajectory, to find which it is required to examine the complete picture of the motion of the pendulum with elastic couplings and a vibrating suspension. In other words, in the formal solution of the problem on translation we have used only two out of the three assumptions, adopted in the analysis of the mechanical model.

In Figure 5.9 we have marked with a solid line the values of translations calculated according to equation (5.23) with the same values of G, R, ρ_0, and α_0, with which the translation was calculated with equation (5.19) (broken line.

Estimations according to both equations yield sufficiently close results, in spite of the fact that these equations were obtained by two basically different methods.

Now we shall turn to the determination of the value of disalignment of the pendulum with elastic couplings, operating under the conditions of vibration of its suspension point and as we did for translation, compare the results yielded by the two different methods of analysis. /182

Returning to the mechanism presented in Figure 5.8, we see that with a harmonic vibration of slider 1, rod 2 will perform forced oscillations. The current angle of the small turn of the rod with respect to the "middle" position, is equal to

$$\varepsilon \approx \sin\varepsilon = \varrho\frac{\sin\alpha_0}{R}.$$

Under the harmonic law of actuation $\rho = \rho_0 \cos \omega t$, and therefore

$$\varepsilon = \frac{\varrho_0 \sin\alpha_0}{R}\cos\omega t. \qquad (5.24)$$

Now, using (5.15) we set up the equation of forced oscillations, and,

bearing in mind that $c_{-2} \approx c_2 << c_0$, we shall compute only the central coefficient c_0.

Here, the amplitude of the first harmonic the period of which is equal to the period of actuation, is found approximately equal to

$$\frac{Q_{20}}{\alpha_0 - 4}.$$

The amplitudes of the second, and all the subsequent harmonics for this instance are small in comparison with the amplitude of the first harmonic, since the forced movement of the mechanism will be performed approximately according to the harmonic law

$$y_{\text{part}} \approx \frac{Q_{20}}{\alpha_0 - 4} \cos 2x. \tag{5.25}$$

According to (4.49) value Q_{20} in expression (5.25) is equal to

$$Q_{20} = 4\rho_0\Theta_{20}.$$

The actuation coefficient for the mechanism examined is (see § 4.5)

$$\Theta_{20} = -\frac{\sin\alpha_0}{R}.$$

Substituting the values of Q_{20} and Θ_{20} indicated in expression (5.25) we get

$$y_{\text{part}} \approx \frac{4\rho_0 \sin\alpha_0}{R(4-\alpha_0)} \cos 2x.$$

Bearing in mind that according to the designations adopted $y_{\text{part}} = \varepsilon$, /18
$2x = \omega t$, we shall get ultimately

$$\varepsilon \approx -\frac{\rho_0 \sin\alpha_0}{R\left[1-\left(\frac{\omega_{n0}}{\omega}\right)^2\right]} \cos\omega t. \tag{5.26}$$

In accordance with this equation, the disalignment of the pendulum attains the value of

$$E = -\frac{2\rho_0 \sin\alpha_0}{R\left[1-\left(\frac{\omega_{n0}}{\omega}\right)^2\right]}.$$

Comparing (5.24) and (5.26) it is easy to find that when $a_0 << 1$ the results of the formal solution of the equation of forced oscillations agree sufficiently well with the illustrative deductions obtained on the basis of the examination of the model. For the case of the free pendulum (i.e., without elastic couplings) located in the horizontal plane, equations (5.24) and (5.26) coincide fully.

§ 5.8. Experiments with a Hinged Four-Component System.

The number of experimental investigations directly or indirectly connected with the problems examined, is not large. In work [108] the results are given of experiments with pendulums, moving in the field of centrifugal forces. These experiments enabled us to determine experimentally the motion corresponding to the periodical solutions of the Mathieu equation, and verify the existence of instability regions. In work [116] the results of experiments are given, connected with the investigation of the parametric resonance; these experiments were also performed with pendulums. Finally, in [34] it has been demonstrated experimentally that under the conditions of vibration of the suspension point the unstable equilibrium position of the pendulum is found to be stable.

At the same time we do not know of a single work in which problems would be posed on the experimental investigation of the effect of vibration of the stand or the pulsation of an external force on a more or less complex system, for example such as a mechanism with elastic couplings.

In the present paragraph we give the results of experiments carried out with a model of a hinged four-component system with elastic couplings, the principal task of these experiments being the solution of the problem of the correctness of the results obtained on the basis of linearization of the differential equations of motion on the mechanism, and are the factors accounted for in the analysis really decisive. /184

For solving this problem it is apparently sufficient to carry out the investigation for practically the most important case, when the stand of the mechanism performs a reciprocating vibration. Here, the analogy in the computation relationships enables us to generalize the obtained deductions for the other cases of actuation, the sources of which are the vibrations of the stand or the pulsations of an external force.

Analyzing the structure of equation (5.4) it is not hard to formulate the general demands to the apparatus intended for experimental verification of the conclusions obtained as the result of its solution.

The experimental apparatus should provide the feasibility of: 1) an independent variation of the amplitude and frequency of actuation, so as to change independently the parameters a_0 and q_{20} of the equation with the invariable parameters and position of the mechanism undergoing tests; 2) changing within a wide range the position of the mechanism tested with respect

to the direction of the vibration so as to vary widely the values Ψ_2 and Θ_2 with unchanged parameters and position with respect to the stand; 3) varying the position of the mechanism with respect to the stand; 4) recording the most important parameters investigated.

The experiments described in the present paragraph, as well as the other experiments described in the subsequent chapters of the present work, were carried out on a special vibration table, which enabled us to regulate smoothly and independently the amplitude and frequency of the vibration. The design and the principle of the action of this vibration table are described in Par. 1 of the Appendix.

An experimental hinged four-component system is shown in Figure 5.10. It consists of stand 1, crank 2, connecting rod 3, and rocker 4. Stand 1 is equipped with a flange, by means of which it is fastened to table 8 of the vibrator with bolt 7, and a slot with respect to which column 5 may move. Into crank 2 a lag screw 9 is screwed in, into the slot of which a small beam 6 is welded. The other end of the beam is introduced into the slot of column 5. /185

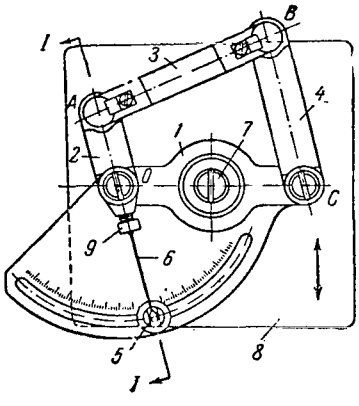

Figure 5.10.

Beam 6 performs two functions at once. It is the elastic coupling imposed on the mechanism, and at the same time, owing to the wire pickups glued onto it, it is the pickup of the movement of the mechanism.

The design of the four-component mechanism ensures the feasibility of: a) orienting it in an arbitrary manner with respect to the direction of the vibration. For this it suffices to loosen screw 7 and install the stand of the mechanism at a prescribed angle to the pointer indicating the direction of the vibration; b) varying, within sufficiently wide limits the position of the mechanism with respect to the stand. For this is suffices to unscrew the nut, holding column 5. In this case the crank with the rod and column rotate freely with respect to point).

The mobile parts of the mechanism, after being manufactured were measured and weighed. The respective results are given in Table 5.1, the designations in the table corresponding to the designations in Figure 5.11.

The values of the derivatives $(d\alpha_3/d\alpha_2)$, $(d\alpha_4/d\alpha_2)$, $(ds_3/d\alpha_2)$ and the inertial coefficient of the mechanism were calculated by two methods: graphical and graphoanalytical. In order not to encumber the text the calculation formulas and the computations connected with them were placed in Par. 2 of the Appendix. The curves of these functions are given in Figures 5.12, 5.13 and 5.14.

Table 5.1. PARAMETERS OF THE MECHANISM

Number of components	Component	l, мм	l_s, мм	G, grams	J_s, g · cm · sec^2	Remark
1	stand	60	—	—	—	—
2	crank	25	11	5	0,023	J_a=0,035
3	connecting rod	60	30	8,4	0,036	—
4	rocker	50	20	7,5	0,055	J_c=0,160

Tr. Note: Commas indicate decimal points.

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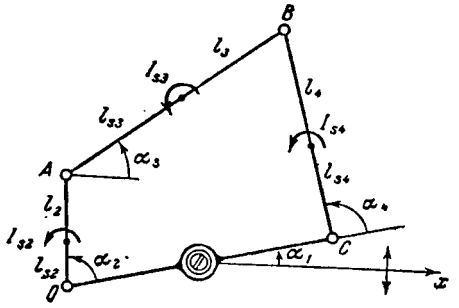

Figure 5.11.

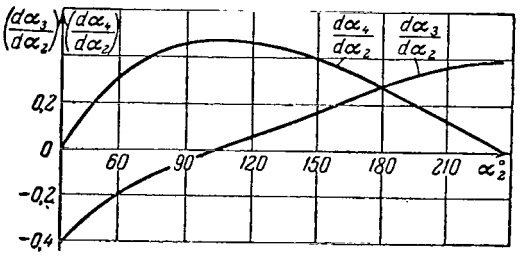

Figure 5.12.

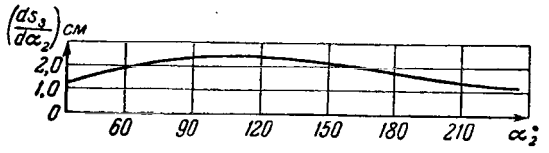

Figure 5.13.

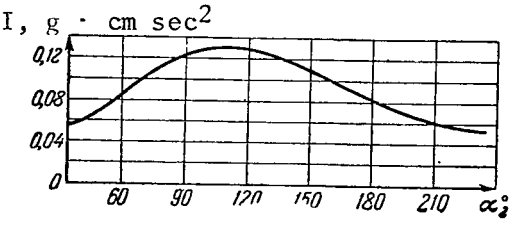

Figure 5.14.

To determine the quasielastic coefficient of the mechanism let us examine Figure 5.15, in which we show the crank with the beam and two possible /187 variations of fastening the beam.

It is not hard to show that for the variations of clamping of the beam, given in Figures 5.15 a and b, the quasielastic coefficient is respectively equal to

$$K_a = \frac{3EJa^2}{(a-b)^3}, \qquad K_b = \frac{3EJb^2}{(a-b)^3}.$$

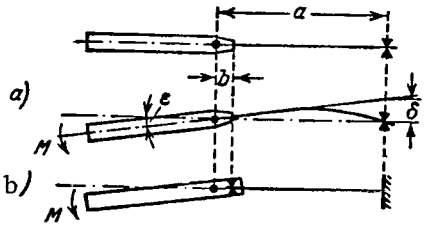

Figure 5.15.

From design considerations and also in order that with the minimum rigidity of the beam we may obtain a sufficiently large value of the quasielastic coefficient, we selected the variation of fastening given in Figure 5.15,a.

With the structural dimensions adopted and the material of the beam the quasielastic coefficient was found to be equal to

$$K = 4{,}500 \text{ g} \cdot \text{cm/rad}.$$

According to the calculated values of the inertial and quasielastic coefficients we plotted the calculation characteristic of frequency ω_n of the free oscillations of the mechanism (solid line in Figure 5.16). Then we performed an experimental checkup of the characteristic (see the oscillograms of the free oscillations in Figure 5.17 and the broken line in Figure 5.16). /188

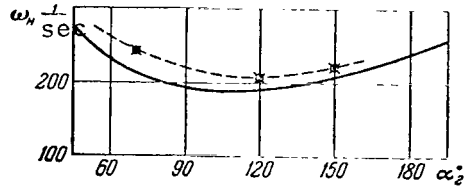

Figure 5.16.

As it should have been expected, in this case the actual frequencies of the mechanism's own oscillations were found to be somewhat higher than the calculated frequencies of the free oscillations. Evidently, the increase of the rigidity of the beam through the pickups glued on it, produces a substantially greater effect on the frequency of the mechanism's own oscillations than friction in the kinematic pairs.

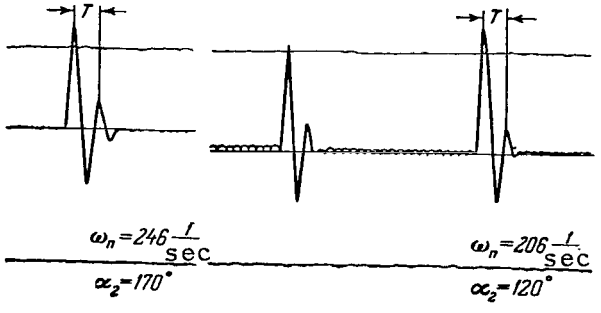

Figure 5.17.

Finally, according to the formulas given in Par. 2 of the Appendix we calculated the coefficients of actuation and periodicity of the mechanism. As it was indicated above, the values of these coefficients depend on both the position of the mechanism with respect to the stand and on the orientation of the stand with respect to the direction of the vibration. The results of the calculation are given in Figures 5.18 and 5.19.

Now we shall return to equation (5.4) of the motion of the mechanism operating under the condition of vibration of the stand. After determining the value ω_n and coefficients Ψ_2 and Θ_2 of the mechanism examined in this /189

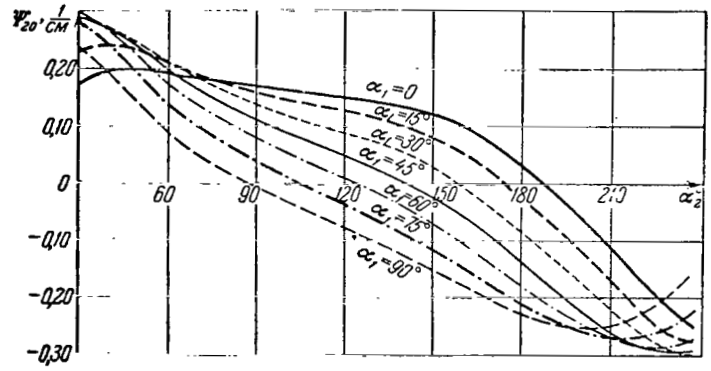

Figure 5.18.

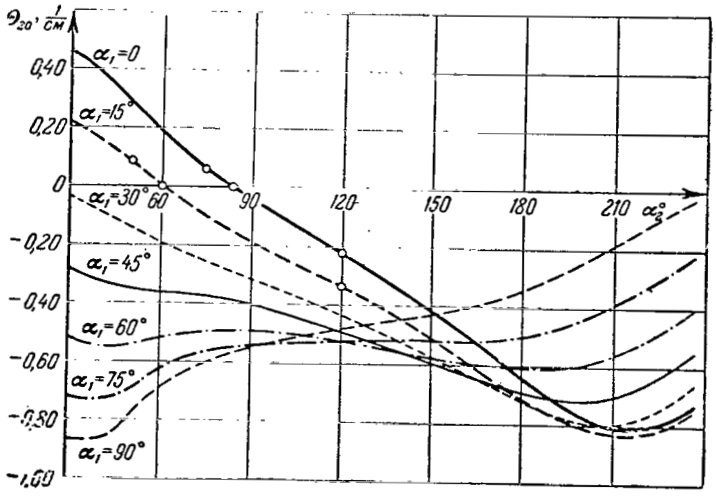

Figure 5.19.

equation, two unknown values ρ_0 and ω remain.

The registering devices/190 provided for in the design of the vibration table enable us to determine the amplitude and frequency of the actuation and consequently to determine the numerical values of all the parameters included in this equation, and then by the previously-described methods to find its solution, and determine the approximate evaluation of the dynamic errors of the mechanism, i.e., to establish the approximate values of the translation and disalignment. However, in the present paragraph we shall do otherwise.

Instead of repeating the previously-performed computations, let us turn our attention to the fact that the analysis of the curves of the actuation coefficients Θ_2, given in Figure 5.19, indicates the method of direct verification of the equations of motion set up and the method of dynamic calculation developed. According to these curves, with certain positions of the mechanism, the actuation coefficient $\Theta_2 = 0$. In this case the right member of equation (5.4) turns to zero, and consequently, the mechanism must remain at rest independently of the intensity of vibration. Let us examine the results of the experiment.

In Figure 5.20 we give the typical oscillogram recorded in the process of experiments with a hinged four-component system. Here, the middle curve represents the forced oscillations of the mechanism, and the upper and lower curve, the displacement and speed of the stand in the process of its vibration.

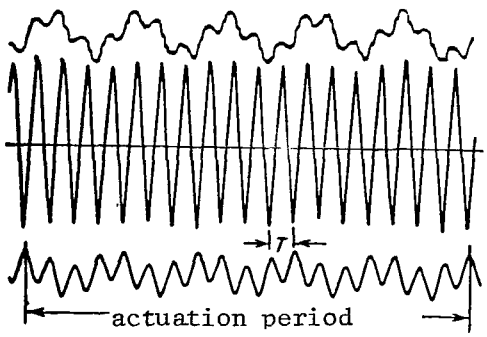

Figure 5.20.

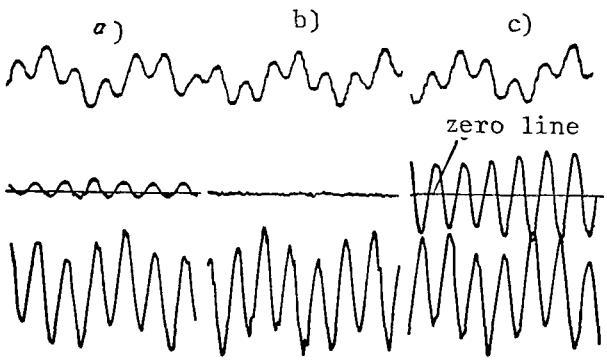

Figure 5.21.

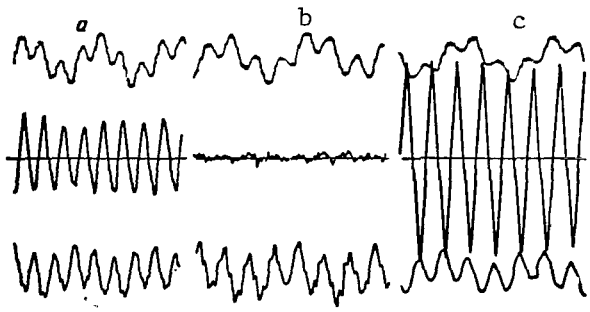

Figure 5.22.

In Figures 5.21 and 5.22 we give /191 the oscillograms of the forced oscillations of the mechanism for different positions and different orientation of it with respect to the direction of the vibration.

The experiment showed that in the above-mentioned positions of the mechanism ($\Theta_2 = 0$) the amplitude of the forced oscillations in reality proves insignificantly small (oscillograms in Figures 5.21,b and 5.22,b). With the increase of coefficient Θ_2 the amplitude of the forced oscillations increases. Here, the character of the forced motion of the mechanism, as it should have been expected, is determined by the /192 actuation component which possesses the maximum intensity, namely, the high-frequency component.

The qualitative coincidence of the expected and obtained results may serve as the best proof of the correctness of the method of calculation of the dynamic precision of the mechanism.

In all the oscillograms the "zero" line was recorded, characterizing the position of the static equilibrium of the mechanism. The measurement of the areas, bounded by the forced oscillation curve, required for determining the translation, may be carried out only approximately. An approximate measurement showed that in actual practice the translation of the mechanism is equal to zero. The same result yields a calculation according to the translation formula which verifies once again the conclusion made in the preceding paragraph; the absence of translation is explained by the relatively small value of the inertial coefficient with a substantial rigidity of the elastic couplings. We should note that the translation phenomenon was

observed upon removal of the elastic couplings, although the absence of elastic couplings excluded the possibility of registering the results of the experiment.

CHAPTER 6

EFFECT OF FRICTION IN KINEMATIC PAIRS

§ 6.1. Estimation of Friction.

Up to the present we carried out the investigation assuming that friction /193
in kinematic pairs of a mechanism, as well as the elastic friction in the material of the elastic couplings is so insignificant, that its effect on the movement of the mechanism with elastic couplings may be disregarded.

In a number of instances this assumption proves quite acceptable and the results of the experiment coincide well with the results of the theoretical analysis.

However, when the mechanism contains a large number of kinematic pairs, special daming devices etc., friction may have a substantial effect on the movement of the mechanism.

Estimation of friction in solving a dynamic problem on the motion of a mechanism is connected with great difficulties. The magnitudes of the forces arising in kinematic pairs depend on the sizes of the reaction, while to determine the latter it is required to solve the problem of the motion of the mechanism. But for solving the problem of the motion of the mechanism it is required to know all the forces acting on its components, and consequently, also the frictional forces.

All this leads us to use the method of consecutive approximations in solving problems requiring an estimation of the frictional forces. Here, the analysis carried out by us earlier, represents only the first step, as the result of which we determine the law of motion of a mechanism not loaded with frictional forces.

The next step is the kinetostatic calculation, the purpose of which is reduced to the determination of reactions in the kinematic pairs of the mechanism.

Knowing the values of the reactions we can pass on to the determination /194
of the forces and moments of frictional forces, acting on the components of the mechanism, and to setting up the differential equation of motion in the second approximation.

Here it is necessary to distinguish between two possible instances. It might be found that in the process of small oscillations of the mechanism, the reactions in the kinematic pairs change so insignificantly, that in the determination of the frictional forces, these reactions may be considered constant. If, in addition, the friction in the kinematic pairs may be considered linear,

then the exact solution of the dynamic problem in the second approximation, i.e., with an allowance for friction, presents no difficulties in principle.

In the other case, i.e., when during the period of the mechanism's oscillations, the values of the reactions in kinematic pairs vary substantially, the problem suddenly becomes more complicated, as the result of the fact that the correlated moment of the frictional forces proves to be a non-linear function of the generalized coordinate and its derivative. Here, the differential equation of motion proves non-linear, as a rule it is impossible to find its exact solution, and for solving this problem in the second approximation we usually have to resort to methods of approximate or numerical integration.

In the present chapter we shall examine the analysis of both instances mentioned. In the first paragraphs we shall examine the problems of the effect of friction in kinematic pairs on the dynamic stability and dynamic precision of the mechanism, as applied to the instance when the values of the reaction are constant in magnitude while the frictional forces are proportional to the relative velocity of the elements of the pairs.

Then we shall pass over to the examination of the instances when the estimation of friction in kinematic pairs results in non-linear equations of motion. Here, instead of using the methods of approximate analysis of similar equations, we shall turn to the examination of simple models, which give us a visual qualitative, and in some cases also a quantitative concept of the possible effect of the action of frictional forces on the motion of a mechanism with elastic couplings. Later we shall use the results obtained for solving problems connected with the dynamic precision of mechanisms.

§ 6.2. The Dissipative Coefficient of a Mechanism.

We shall skip all the reasonings and computations connected with the usual kinematic calculation of a mechanism, the law of motion of which, in the first approximation, is known to us. Let us assume that as the result of this calculation, the reactions in all the kinematic pairs for a certain "middle" position (α_{dyn}) of the mechanism (Figure 6.1). /195

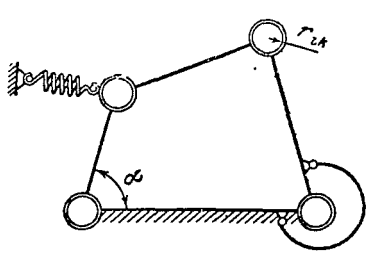

Figure 6.1.

Let the frictional forces in the kinematic pairs be proportional to the first degree of the relative velocity of the elements of the kinematic pair, in such a way that

$$F_i = \beta_i v_i, \quad M_j = \beta_j \omega_j r_j, \tag{6.1}$$

where F_i is the frictional force in the i-th advancing kinematic pair, M_j is the moment of the frictional forces in the j-th rotating kinematic pair, r_j is the radius of the pivot or bearing

of the kinematic pair, β is the proportionality coefficient, determined in the assumption that the reaction in the kinematic pair remains constant during the entire process of motion. v_i, ω_j are moduli of the relative velocities of the elements of kinematic pairs.

Having determined, as the result of approximate calculation, the position of the dynamic equilibrium α_{dyn}, we can formulate an expression for the generalized moment of friction in the following form:

$$M_{fr} = -\left[\sum_{i=0}^{m} F_i \left(\frac{ds_i}{d\alpha}\right)_{dyn} + \sum_{j=0}^{n} M_j \left(\frac{d\alpha_j}{d\alpha}\right)_{dyn}\right], \qquad (6.2)$$

where m is the number of advancing kinematic pairs, n is the number of rotating kinematic pairs.

Having designated as before the coordinate characterizing the deviation of the mechanism from position α_{dyn}, with ε and keeping in mind that

$$v_i = \left(\frac{ds_i}{d\alpha}\right)_{dyn}\dot{\varepsilon}, \qquad \omega_j = \left(\frac{d\alpha_j}{d\alpha}\right)_{dyn}\dot{\varepsilon},$$

we can write equation (6.2) in the following way /196

$$M_{fr} = -\dot{\varepsilon}\left[\sum_{i=0}^{m} \beta_i \left(\frac{ds_i}{d\alpha}\right)^2_{dyn} + \sum_{j=0}^{n} \beta_j r_j \left(\frac{d\alpha_j}{d\alpha}\right)^2_{dyn}\right].$$

We shall call the expression of the following type:

$$B = \frac{1}{J_{dyn}}\left[\sum_{i=0}^{m} \beta_i \left(\frac{ds_i}{d\alpha}\right)^2_{dyn} + \sum_{j=0}^{n} \beta_j r_j \left(\frac{d\alpha_j}{d\alpha}\right)^2_{dyn}\right].$$

the dissipative coefficient B of the mechanism.

The dissipative coefficient of the mechanism yields a quantitative concept of the dissipative properties, acquired by the mechanism as the result of the effect of friction. It depends on the coefficients of friction in individual kinematic pairs, on the dimensions of these kinematic pairs, and also on the position and configuration of the mechanism.

Using the concept of the dissipative coefficient of the mechanism we can set up an equation of the mechanism's own oscillations* in the following form:

* For convenience of presentation, the mechanism's own oscillations, in contrast to its free oscillations, will signify the small attenuating oscillations of the mechanism, occurring under the effect of elastic forces, weight, and dissipative forces.

$$\ddot{\varepsilon} + B_0\dot{\varepsilon} + \omega_{n0}^2\varepsilon = 0, \tag{6.3}$$

where B_0 is the value of the dissipative coefficient of the mechanism in the position of static equilibrium, ω_{n0} is the frequency of the free oscillations with respect to the static equilibrium position.

Using the conventional methods for the analysis of equations of type (6.3), we could determine the logarithmic decrement of attentuation of oscillations, establish the character of the effect of configuration and position of the mechanism on the frequency of its own oscillations, etc. However, the reasonings and computations connected with such an analysis are sufficiently evident, while the problems of solution of type (6.3) equations were already examined in Chapter 2. Considering it superfluous to repeat them here, we shall now examine the effect of friction on the stability of the mechanism and its forced motion.

§ 6.3. Effect of Linear Friction on the Dynamic Stability of the Mechanism.

Let the stand of the mechanism presented in Figure 1, perform a reciprocating motion according to the harmonic law. Then, the differential equation of the mechanism's motion, with respect to the vibrating stand, disregarding the effect of frictional forces (i.e., in the first approximation) has the form of (4.33). /197

Introducing into the expression of the generalized moment of the mechanism a term accounting for the effect of linear friction, setting up Lagrange's equation, and carrying out the computations, analogous to the ones performed in Chapter 4, we get the equation of the forced oscillations of the mechanism

$$\ddot{\varepsilon} + B\dot{\varepsilon} + (\omega_{n_{\text{dvn}}}^2 + \varrho_0\omega^2\Psi_2\cos\omega t)\,\varepsilon = d + \varrho_0\omega^2\Theta_2\cos\omega t. \tag{6.4}$$

Written in the canonical form, (see § 4.6), this equation assumes the form of

$$\frac{d^2y}{dx^2} + 2\sqrt{f}\frac{dy}{dx} + (a - 2q_2\cos 2x)\,y = Q_0 + Q_2\cos 2x, \tag{6.5}$$

where

$$\sqrt{f} = \frac{B}{\omega}.$$

For the convenience of the further analysis we shall transform equation (6.5) in such a way as to get rid of the member containing dy/dx. Here, the auxiliary function v of the linear transformation y = zv, according to (6.5), must have the following appearance (see [71]):

$$v = e^{-\sqrt{f}x}.$$

Having substituted the new dependent variable z into (6.5) we get

$$z'' + (\bar{a} - 2q_2 \cos 2x) z = e^{\sqrt{f}x}(Q_0 + Q_2 \cos 2x), \tag{6.6}$$

where $\bar{a} = a - f$ is the frequency of the intrinsic oscillations of the mechanism with elastic couplings.

The mechanism's equation of motion has a similar appearance when its components are acted upon by a force, pulsing according to the harmonic law.

The problem of the dynamic stability of a mechanism is solved by means of the analysis of stability of the solution of a homogeneous equation equivalent to equation (6.6). Now we know that the solution of this homogeneous equation has the following appearance: /198

$$z = e^{\mu x} \sum_{r=-\infty}^{\infty} c_{2r} e^{2rix} + e^{-\mu x} \sum_{r=-\infty}^{\infty} c_{2r} e^{-2rix}.$$

Returning to the dependent variable y we can rewrite this solution in this way

$$y = e^{(\mu - \sqrt{f})x} \sum_{r=-\infty}^{\infty} c_{2r} e^{2rix} + e^{-(\mu - \sqrt{f})x} \sum_{r=-\infty}^{\infty} c_{2r} e^{-2rix}. \tag{6.7}$$

We remind the reader that in case of the absence of friction (f = 0) solution (6.7) with $x \to \infty$ will remain limited only under the condition that $\mu = i\beta$, where β is a real nonintegral number. Here, the regions of values of parameters a and q, corresponding to stable and unstable solutions, are divided by the lines of characteristic numbers a_0, b_1, a_1, ... , drawn in a dotted line in Figure 6.2. In regions of unstable solutions, the characteristic index represents either a real number or a complex number of the type of $\mu = \alpha + in$, where n = 0, 1, 2, ...

Substituting $\mu = \alpha + in$ in (6.7), we get

$$y = e^{(\alpha - \sqrt{f})x} \sum_{r=-\infty}^{\infty} c_{2r} e^{(2r+n)ix} + e^{-(\alpha - \sqrt{f})x} \sum_{r=-\infty}^{\infty} c_{2r} e^{-(2r+n)ix}.$$

Upon examination of this equality it becomes obvious that the presence of friction in kinematic pairs does not exclude the possibility of origination of dynamic instability of the mechanism. The instability may arise under the condition that the real part of the characteristic attains the value of /199

$$R(\mu) = \alpha \geqslant \sqrt{f}.$$

The sets of those points of the stability "chart", for which the equality $\alpha = \sqrt{f}$ exists, form new boundaries, separating the regions of stable and unstable solutions in the presence of friction.

The value of the characteristic index μ is determined by the values of parameters q and a or $\bar{a}$ in equations (4.50) or (6.6). Consequently, the problem of determination of the limits of stability is reduced to calculation of the values of the parameters, with which

$$R(\mu) = \alpha = \sqrt{f} = \text{const.}$$

The method and results of the calculations indicated are given in work [98].

In Figure 6.2 we have drawn the lines of the characteristic figures for various values of the attenuation factor f. As we can see, in the presence of friction, the regions of stable solutions prove wider, while the regions of unstable solutions are narrower in comparison with the regions, into which the stability "chart" is divided, if we do not take friction into account. With the increase of the dissipative coefficient of the mechanism, the boundaries of the regions of stable solutions expand more and more, while the regions of unstable solutions narrow down continuously. We should note that here, with the same value of the dissipative coefficient, the narrowing of the limits of unstable solutions is found to be different for various regions. For the second region of instability, even a small attenuation has an appreciable effect (compare lines f_1 and f_4). In the third region of instability this effect is still more appreciable, etc. For this reason, in real structures the instability phenomenon may arise with a low excitation frequency (which corresponds to the second, third, and other instability regions) only with especially unfavorable values of the mechanism's parameters, a relatively large vibration amplitude, and low values of the attenuation factor.

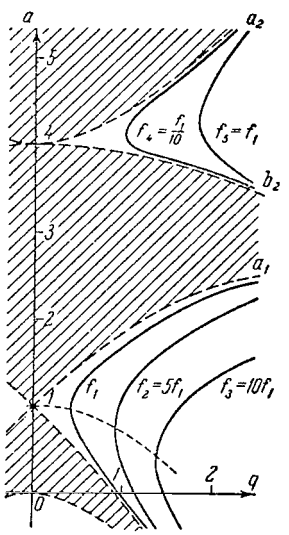

Figure 6.2.

§ 6.4. Resonance and Dynamic Errors of the Mechanism under Linear Friction Conditions.

If the problem of the dynamic stability of the mechanism is solved on the /200 basis of the analysis of the general solution of a homogeneous equation, in order to establish the conditions of origination of the resonance and to clear up the question of the effect of friction on the dynamic precision of the mechanism, it is required to resort to examination of a particular solution of equation (6.5), which has the following appearance:

$$y = \frac{Q_0}{\delta}\left[\sum_{r=-\infty}^{\infty} c_{2r} e^{-2rix} \sum_{r=-\infty}^{\infty} \frac{c_{2r}}{2ri+\mu+\sqrt{f}} e^{2rix} + \right.$$
$$\left. + \sum_{r=-\infty}^{\infty} c_{2r} e^{2rix} \sum_{r=-\infty}^{\infty} \frac{c_{2r}}{2ri+\mu+\sqrt{f}} e^{-2rix}\right] +$$
$$+\frac{Q_2}{2\delta}\left[\sum_{r=-\infty}^{\infty} c_{2r} e^{-2rix} \sum_{r=-\infty}^{\infty} \frac{c_{2r}}{2(r+1)i+\mu+\sqrt{f}} e^{2(r+1)ix} + \right.$$
$$+ \sum_{r=-\infty}^{\infty} c_{2r} e^{2rix} \sum_{r=-\infty}^{\infty} \frac{c_{2r}}{2(r-1)i+\mu+\sqrt{f}} e^{-2(r-1)i} +$$
$$+ \sum_{r=-\infty}^{\infty} c_{2r} e^{-2rix} \sum_{r=-\infty}^{\infty} \frac{c_{2r}}{2(r-1)i+\mu+\sqrt{f}} e^{2(r-1)ix} +$$
$$\left. + \sum_{r=-\infty}^{\infty} c_{2r} e^{2rix} \sum_{r=-\infty}^{\infty} \frac{c_{2r}}{2(r+1)i+\mu+\sqrt{f}} e^{-2(r+1)ix}\right]. \qquad (6.8)$$

The analysis of this solution indicates that in systems, the motion of which satisfies linear equations with periodic coefficients, the unlimited increase of the amplitude is possible even in the presence of dissipative forces.

It is obvious, that particular solution (6.8) remains limited when $x \to \infty$ only under the condition that

$$2(r \pm 1)i + \mu \pm \sqrt{f} \neq 0.$$

In the case when

$$2(r \pm 1)i + \mu \pm \sqrt{f} = 0, \qquad (6.9)$$

the coefficient at the corresponding term of particular solution (6.8) turns into infinity. /201

Substituting in (6.9) the value of the characteristic index, in the general case equal to $\mu = \alpha + \beta i$, we get the following conditions of origination of resonance:

$$i[2(r \pm 1) \pm \beta] = 0; \qquad \alpha \pm \sqrt{f} = 0.$$

From the examination of these conditions it is not hard to determine, that the resonance may begin only in the region of unstable solutions, with such values of parameters $\bar{a}$ and q, with which

$$\beta = 2n,$$

where n is an integral number.

Evidently, the curves, connecting those points of the stability "chart", for which the resonance conditions are fulfilled, are located in even-numbered regions of unstable solutions of the homogeneous equation, corresponding to equation (6.6). Here the characteristic index is

$$\mu = \sqrt{f} + 2ni.$$

Thus, in the case of fluid friction, just as also in the case of the absence of friction, resonance arises with values of the parameters, corresponding to the boundaries between the regions of stable and unstable solutions of the homogeneous equation, corresponding to equation (6.5).

Now, let us examine the problem of the effect of frictional forces in kinematic pairs on the value of the amplitude of oscillations, and the dynamic precision of the mechanism. For this it is sufficient to establish the character of the effect of frictional forces on the withdrawal and erosion of the mechanism.

Using the earlier-indicated method, we shall separate from the particular solution (6.8) a constant term H_0, equal to

$$H_0 = \frac{Q_0}{\delta} H_0^{Q_0} + \frac{Q_2}{\delta} H_0^{Q_2}.$$

In this equation, coefficients $H_0^{Q_0}$ and $H_0^{Q_2}$ are respectively equal to /202

$$\left.\begin{aligned} H_0^{Q_0} &= \sum_{r=-\infty}^{\infty} c_{2r}^2 \left[\frac{1}{2ri+\mu+\sqrt{f}} + \frac{1}{2ri+\mu-\sqrt{f}} \right], \\ H_0^{Q_2} &= \sum_{r=-\infty}^{\infty} c_{2r} \left[\frac{c_{2(r-1)}}{2(r-1)i+\mu-\sqrt{f}} + \frac{c_{2(r-1)}}{2(r+1)i+\mu+\sqrt{f}} \right]. \end{aligned}\right\} \qquad (6.10)$$

Assuming that the dynamic stability of the mechanism is assured, we find the following expression for coefficient $H_0^{Q_0}$

$$H_0^{Q_0} = -2i \sum_{r=-\infty}^{\infty} c_{2r}^2 \frac{2r+\beta}{f+(2r+\beta)^2}.$$

Keeping (5.14) in mind, we determine the imaginary part of coefficient $H_0^{Q_2}$, equal to

$$-i \sum_{r=-\infty}^{\infty} c_{2r} \left\{ \frac{c_{2(r+1)}[2(r+1)+\beta]}{f+[2(r+1)+\beta]^2} + \frac{c_{2(r-1)}[2(r-1)+\beta]}{f+[2(r-1)+\beta]^2} \right\}.$$

As we already know, with small values of withdrawal, its value in the instance of the reciprocating harmonic vibration of the stand is determined from the following equation (see (5.20)):

$$\Delta_2 = \frac{H_0^{Q_{20}}}{H_0^{Q_0}} \frac{\Theta_2}{K} \varrho_0 \omega^2 J.$$

The attenuation factor f, according to (6.10) is included in the numerator and denominator of this expression. In addition to this, the attenuation factor figures in equations, from which we determine coefficients c_{2r}, dependent on the value $\bar{a} = a - f$. On the basis of this we can draw the conclusion that the presence of friction in kinematic pairs affects in some degree the value of withdrawal of the mechanism.

However, the quantitative evaluation of this effect may be obtained only taking into account the concrete values of the parameters of the mechanisms and parameters of excitation.

As an example we shall use again the practically important instance, examined by us in detail, when the intensity of the excitation is sufficiently great ($a << 1$), and the unbalance of the mechanism is not large ($q_{20} << 1$). /203

As before, we limit ourselves to estimation of only three central coefficients c_{-2}, c_0, c_2. Then according to (6.10), we get

$$H_0^{Q_0} \approx -2i\beta \frac{c_0^2}{f + \beta^2}; \qquad H_0^{Q_{20}} \approx -i\beta \frac{c_0 c_2}{f + \beta^2},$$

where

$$\beta = -\sqrt{\bar{a}}.$$

Consequently,

$$\frac{H_0^{Q_{20}}}{H_0^{Q_0}} \approx \frac{c_2}{c_0} = -\frac{q_{20}}{4}.$$

Thus, in the instance examined, frictional forces exercise no influence on the value of withdrawal of the mechanism. This result becomes understandable if we take into account that when $a << 1$ and $q_{20} << 1$ the forced oscillations of the mechanism have an almost harmonic character (see equation (5.25)). Here, the picture of the effect of the generalized moment of the frictional forces on the mechanism proves symmetrical during the motion of the latter with respect to the dynamic equilibrium configuration. But, with such a symmetrical picture, the mean value of the moment of frictional forces is equal to zero and it cannot affect the withdrawal value.

In the general case, i.e., when the forced oscillations of the mechanism are essentially nonharmonic, the symmetrical picture of the effect of frictional forces is disrupted, as the result of which we may expect that the dynamic equilibrium configuration will change.

We shall not examine in detail the problem of the effect of the forces of linear friction on the value of the second component of the dynamic error. It suffices to state that in this respect the picture is analogous to that, which exists for the systems satisfying the equations with constant coefficients. Under conditions which are far from the resonance conditions, the forces of fluid friction have little effect on the amplitude of the forced oscillations, and consequently, on the value of erosion of the mechanism. /204
With the approach to the resonance conditions, the influence of the frictional forces becomes more substantial, and then, in order to obtain a qualitative evaluation of this effect, it is necessary to use the above-adduced relationships.

§ 6.5. Effect of Dry Friction. Reciprocating Vibration of the Stand.

In the preceding paragraphs we examined the problems of dynamic analysis of a mechanism with elastic couplings in the assumption that the reactions in the kinematic pairs do not change in the process of motion of the mechanism. An approximate estimate of the effect of frictional forces enabled us to linearize the dynamic problem and obtain its solution.

The character of this solution and certain most important properties of mechanisms with elastic couplings were examined applicably to the case of linear friction. Similarly, in the case of dry friction the condition of constancy of the values of the reaction in the process of the motion of the mechanism makes it possible to find the solution to the problem with relatively simple methods. At the same time, we should mention that the structure of the dissipative coefficient of the mechanism, in the case of dry friction (B_c) will differ from the structure of the earlier-determined dissipative coefficient B.

We can set up the expression for B_c, proceeding from the following considerations. The dry friction forces and the moments of these forces arising in kinematic pairs of the mechanism, with respect to size are respectively equal to

$$F_{ic} = f_i R_i; \qquad M_{jc} = f_j r_j R_j,$$

where R_i and R_j are magnitudes of the reaction in the advancing and rotary pairs respectively, f_i and f_j are dry-friction coefficients.

Consequently, the generalized moment M_{fr} of the action of dry-frictional forces will have the following appearance

$$M_{fr} = -\left[\sum_{i=0}^{k} f_i R_i \left|\frac{ds_i}{d\alpha}\right|_{dyn} + \sum_{j=0}^{l} f_j r_j R_j \left|\frac{ds_j}{d\alpha}\right|_{dyn}\right] \operatorname{sign} \dot{\varepsilon}.$$

As the dissipative coefficient B_c of the mechanism, in the case of dry friction, we shall understand the following value /205

$$B_c = \frac{1}{J_{dyn}}\left[\sum_{i=0}^{k} f_i R_i \left|\frac{ds_i}{d\alpha}\right|_{dyn} + \sum_{j=0}^{l} f_j r_j R_j \left|\frac{d\alpha_j}{d\alpha}\right|_{dyn}\right].$$

Then, by analogy with (6.3), it is not difficult to set up the differential equation of the mechanism's own oscillations, which will have the following appearance

$$\ddot{\varepsilon} + \omega_{n_0}^2 \varepsilon = -B_c \operatorname{sign} \dot{\varepsilon}. \tag{6.11}$$

The procedure for solving this equation and the principal properties of the system, the motion of which satisfies it, are sufficiently well known (see, for example, [28]).

As indicated earlier, the problem becomes appreciably more complex in instances when the values of the inertial components of the reactions in the kinematic pairs prove comparable with the values of static components.

Then their influence on the value of frictional forces cannot be ignored, and the latter are found dependent on the generalized coordinate and its derivative.

In the present and the following praragraphs we shall examine several instances, which enable us to form a concept of the effect of dry friction on the behavior of the mechanism, operating under conditions of flat vibration of its stand. In particular, these examples yield an illustrative concept of the causes, through which dry friction in kinematic pairs may serve as the source of dynamic errors of the mechanism.

Let us examine the simplest, ideally-balanced mechanism with elastic couplings, presented in Figure 6.3, the stand of which is performing reciprocating vibrations.

Let the direction of the static component of the reaction in the bearing O coincide with axis y, the vibration of the stand being performed in the same direction, according to the harmonic law. We designate the static component of the reaction with R_{st}. Then, if the motion of the stand is performed according to the law /206

$$y = \varrho_0 \cos \omega t,$$

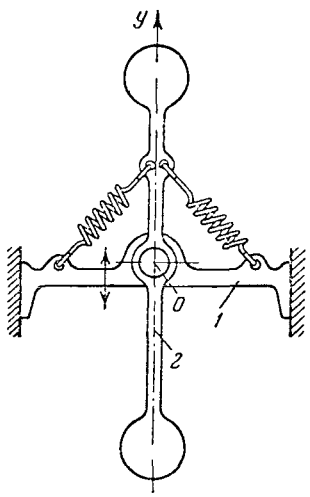

Figure 6.3.

the maximum value of the inertial component of the reaction is

$$R_{in}=M\varrho_0\omega^2.$$

Designating with κ the ratio, characterizing the intensity of the inertial load

$$\varkappa=\frac{M\varrho_0\omega^2}{R_{st}},$$

we can write the instantaneous motion of the dynamic reaction in the following way:

$$R_{dyn}=R_{st}(1-\varkappa\cos\omega t).$$

In this way, the dynamic reaction varies periodically with respect to its value, and when $\kappa > 1$, even with respect to direction. This constitutes a certain explanation of the circumstance that under the vibrational conditions, the stagnation phenomena in mechanisms, caused by friction, become substantially weaker.

Let component 2, as the result of initial perturbation, perform an oscillating motion, deviating by a small angle from the state of equilibrium. Here, in the kinematic pair a frictional force of variable value arises, the intensity of the effect of which on the motion of the mechanism may be estimated according to the mean absolute value of the reaction. This mean value may be determined as follows. The mean value of the reaction, during the first quarter of period T of the vibration is equal to

$$R_{dyn}=\frac{4R_{st}}{T}\int_0^{\frac{T}{4}}(1-\varkappa\cos\omega t)\,dt=R_{st}\left(1-\frac{2\varkappa}{\pi}\right). \qquad (6.12)$$

During the second quarter of the period it will be /207

$$R_{dyn}^{2}=R_{st}\left(1+\frac{2\varkappa}{\pi}\right). \qquad (6.13)$$

Now, in order to determine the mean absolute value of the reaction $R_{dyn\ mn}$ during the full vibration period it is sufficient to calculate the arithmetical mean of the absolute values $R_{dyn\ 1}$ and $R_{dyn\ 2}$:

$$R_{\text{dyn mn}} = \frac{|R_{\text{dyn}}^1| + |R_{\text{dyn}}^2|}{2}. \qquad (6.14)$$

Substituting in (6.14) the absolute magnitudes of value (6.12) and (6.13), we get

$$R_{\text{dyn mn}} = \frac{\left|1 - \frac{2\varkappa}{\pi}\right| + \left|1 + \frac{2\varkappa}{\pi}\right|}{2}.$$

Evidently, we can examine the two following instances:

1) $\varkappa \leq \frac{\pi}{2}$; with these values of $\varkappa$ we shall have

$$R_{\text{dyn mn}} = \frac{\left(1 - \frac{2\varkappa}{\pi}\right) + \left(1 - \frac{2\varkappa}{\pi}\right)}{2} R_{st} = R_{st},$$

and consequently, the mean absolute value remains constant and equal to the value of the static reaction.

2) $\varkappa \geq \frac{\pi}{2}$; here, the mean value of the reaction is found equal to

$$R_{\text{dyn mn}} = \frac{\left(\frac{2\varkappa}{\pi} - 1\right) + \left(1 + \frac{2\varkappa}{\pi}\right)}{2} R_{st} = \frac{2\varkappa}{\pi} R_{st},$$

and consequently, the mean value of the reaction increases in proportion to $\varkappa$.

By the method indicated we can determine the mean value of the dynamic reaction in the case of the nonharmonic vibration law. For this it is sufficient to find the mean value of the reaction in the course of the corresponding parts of the full vibration period.

Knowledge of instantaneous and mean values of the reaction in the kinematic pair enables us to determine the instantaneous and mean values of the frictional-force moment arising in the process of relative motion of the kinematic pair's elements. /208

However, we should emphasize the obvious fact that under the conditions adopted in the example examined, this relative motion may arise only as the result of initial perturbation not connected with the presence of friction between the elements of the pair. Now we change the conditions of the problem, in the part pertaining to the direction of the vibration. Let the reciprocating vibration of the stand proceed in the direction which forms angle γ with the y axis (Figure 6.4).

The dynamic reaction in the kinematic pair in the instance examined will

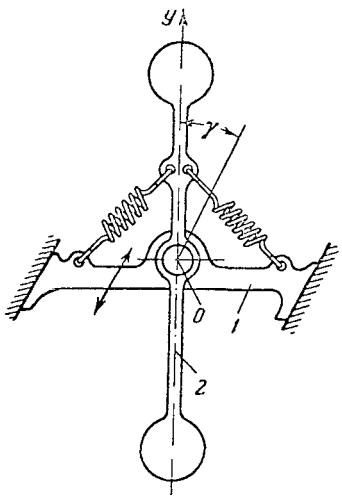

Figure 6.4.

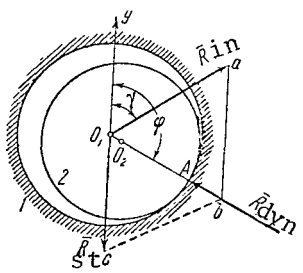

Figure 6.5.

represent a vectorial sum

$$\overline{R}_{dyn} = \overline{R}_{st} + \overline{R}_{in}.$$

We designate with ϕ the current value of the angle between the y axis and the direction of $\overline{R}_{dyn}$ (Figure 6.5).

In the first approximation we can consider that the pivot contacts the bearing in point A of the intersection of the line of action of reaction $\overline{R}_{dyn}$ with the circumference of the bearing. Here, at different time moments, as the result of variation of vector $\overline{R}_{in}$, different points of pivot and bearing will come into contact. Owing to the effect of frictional forces, the pivot will roll on the bearing, while element 2 will perform forced periodic oscillations. Let us determine the amplitude of these oscillations. /209

The instanteneous value of the vector of reaction $\overline{R}_{dyn}$ is determined from the following relationship (see Figure 6.5):

$$R_{dyn}^2 = R_{st}^2 + R_{in}^2 + 2R_{st}R_{in}\cos\gamma. \qquad (6.15)$$

With the harmonic law of vibration, keeping in mind that $\kappa = \frac{M\rho_0\omega^2}{R_{st}}$, we can write (6.15) in the following way:

$$R_{dyn} = R_{st}\sqrt{1+\kappa^2\cos\overline{\omega t} - 2\kappa\cos\gamma\cos\omega t}. \qquad (6.16)$$

Planning the vector triangle Oab on the y axis, we get

$$R_{in}\cos\gamma + R_{st} + R_{dyn}\cos\phi$$

from which

$$\cos\varphi = \frac{R_{st}(\kappa\cos\gamma - 1)}{R_{dyn}}.$$

Substituting the value of R_{dyn} from (6.16), we get

$$\cos\varphi = \frac{\varkappa\cos\gamma - 1}{\sqrt{1 + \varkappa^2\cos^2\omega t - 2\varkappa\cos\gamma\cos\omega t}}. \qquad (6.17)$$

Evidently, angle ϕ reaches the extreme values when $t = 0$ and when $t = \frac{\pi}{\omega}$, while

$$\cos\varphi_1 = \frac{\varkappa\cos\gamma - 1}{\sqrt{1 + \varkappa^2 - 2\varkappa\cos\gamma}}, \qquad \cos\varphi_2 = -\frac{\varkappa\cos\gamma + 1}{\sqrt{1 + \varkappa^2 + 2\varkappa\cos\gamma}}.$$

Knowing the extreme positions of the pivot in the bearing, it is not hard to determine the amplitude of the forced oscillations of component 2, caused by the effect of dry friction.

If we designate the difference between angles $\phi_1 - \phi_2$ with ϕ, and the angle of turn of component 2 corresponding to this difference with E, the relationship between angles E and ϕ is expressed by the following ratio:

$$E = \frac{r}{r_p}\phi, \qquad (6.18)$$

where r is the value of the radial clearance, r_p is the radius of the pivot.

Angle E represents the doubled amplitude of the forced oscillations of component 2, i.e., its erosion, while according to the conditions, $r << r_p$ and consequently, $E << \phi$. /210

Comparing the mean integral value of the coordinate

$$\varphi_{mn} = \frac{1}{\varphi}\int_{\varphi_1}^{\varphi_2}\varphi(t)\,dt$$

with the coordinate characterizing the position of the static equilibrium of the component (in this case, $\phi_{st} = \pi$), we can determine the withdrawal value, equal to

$$\Delta = \phi_{mn} - \phi_{st}.$$

The examples examined indicate that under the conditions of reciprocating vibration, the dry frictional force, generally speaking, may be the cause of dynamic errors of a balanced mechanism. However, the values of the above errors with negligibly small values of the clearances, (this case precisely is being examined) prove negligibly small.

At the same time, from practical work we know, that with a definite character of excitation, the dynamic errors of a balanced mechanism may reach a substantial value even in the presence of negligibly small clearances in

the kinematic pairs. In order to examine this phenomenon, we resort to the analysis of the effect of dry friction forces on the motion of the mechanism, in the case of translational-rotary vibration of its stand.

§ 6.6. The Case of Translational-Rotary Vibration of the Stand.

Just as in the preceding instances, we shall examine the simplest mechanism, representing a balanced pendulum with elastic couplings. First, we shall find out how this pendulum moves, under the condition that the static component of the reaction remains constant in its value and direction, and then determine, what effect on the mechanism's behavior is caused by the circumstance that the R_{st} reaction actually depends on the position of the mechanism and characteristics of the elastic couplings, and varies with its motion.

In the process of rotary vibration, all the points of the mechanism's stand move forward with equal velocity v over circular trajectories, the radius of which will be designated with ρ_0 (Figure 6.6). In the first approximation we will consider that the reaction in the kinematic pair is exactly equal to a certain sum /211

$$\overline{R}_{dyn} = \overline{R}_{st} + \overline{R}_{in},$$

where

$$R_{in} = M\frac{v^2}{\rho_0} = \varkappa R_{st}.$$

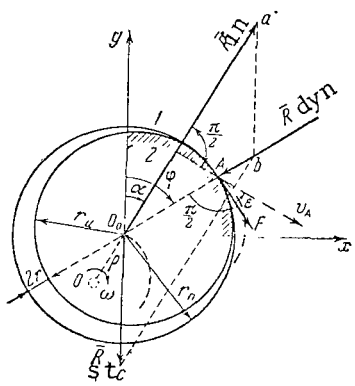

Figure 6.6.

At various time moments, owing to the turn of vector $\overline{R}_{dyn}$ various points of the pivot and bearing will come into contact.

If we exclude the possibility of pure rolling of the pivot on the bearing, then in the process of replacement of the points of contact, the pivot will be acted upon by the frictional force of slipping, drawing it in the direction of the tangential velocity component of the point of contact A (Figure 6.6). The rotating moment of this frictional force, about which we shall, at this time, talk as about the conventional moment of friction, may either coincide with respect to direction, with the angular velocity ω of rotation of vector $\overline{R}_{in}$, or be directed to the opposite side, depending on the direction of the tangential velocity component of point A.

In order to determine the direction of this velocity component we designate with $\alpha = \omega t = v_0/\rho_0\, t$ the angle characterizing the direction of vector $\overline{R}_{in}$, and with ϕ the angle characterizing the direction of the reaction

vector $\overline{R}_{dyn}$.

Then, from triangle O_2ab (see Figure 6.6), we get

$$R_{st}^2 = \kappa^2 R_{st}^2 + R_{dyn}^2 - 2\kappa R_{st} R_{dyn} \cos \varepsilon$$

where

$$\varepsilon = \phi - \alpha.$$

Consequently, /212

$$\cos \varepsilon = \frac{R_{st}^2 (\kappa^2 - 1) + R_{dyn}^2}{2\kappa R_{st} R_{dyn}} \tag{6.19}$$

Keeping in mind that the projection of the dynamic reaction on the axis of the coordinates are equal to

$$R_{dyn\,\overline{x}} = \kappa R_{st} \sin \alpha, \qquad R_{dyn\,\overline{y}} = \kappa R_{st} \cos \alpha - R_{st},$$

we can write the instantaneous value of the reaction vector modulus as follows:

$$R_{dyn} = R_{st} \sqrt{1 - 2\kappa \cos \alpha + \kappa^2}. \tag{6.20}$$

Substituting (6.20) in (6.19), we get

$$\cos \varepsilon = \frac{\kappa - \cos \alpha}{\sqrt{1 - 2\kappa \cos \alpha + \kappa^2}}. \tag{6.21}$$

From this it follows that with values $\kappa \geq 1$, we always have $\cos \varepsilon \geq 0$. Accordingly $\varepsilon \leq \pi/2$, and therefore, the direction of the conventional moment of frictional forces always coincides with the direction of the angular velocity ω of rotation of vector $\overline{R}_{in}$. With values $\kappa < 1$ the direction of the conventional moment of frictional forces in a single full period of the stand's vibration, changes twice, these changes occurring with values $\varepsilon = \pi/2$. According to (6.21) $\varepsilon = \pi/2$ when $\cos \alpha = \kappa$. The value of α determined from this, will be designated with α_κ.

Consequently, when $\kappa < 1$ the conventional moment of frictional forces during a certain period of time acts in one direction, and then in the opposite direction, the periods of its action in the direct and reverse direction, being unequal.

In order to evaluate the possible effect of friction on the mechanism's

behavior, we determine the mean values of the moduli of reactions R_{dyn} for the corresponding time intervals. For this we will use function $R_{dyn} = R_{dyn}(\alpha)$, presented by equation (6.20). Taking into account the approximate character of the analysis, we limit ourselves to an approximate appraisal of value $R_{dyn\ mn}$. Here, we may represent (6.20) in the following way: /213

$$R_{dyn}(\alpha) \approx R_{st}\sqrt{1+\varkappa^2}\left(1-\frac{\varkappa_1}{2}\cos\alpha\right),$$

where

$$\varkappa_1 = \frac{2\varkappa}{1+\varkappa^2} \leqslant 1.$$

Having in view the full symmetry of the picture of motion we can determine the mean value of the reaction modulus by means of examining a single half-period of rotary vibration. Depending on the intensity of vibration, two instances are possible:

1) $\kappa \geq 1$. Here, the conventional moment of frictional forces during the entire period of motion retains the same sign, while

$$R_{dyn\ mn}(\alpha) \approx R_{st}\frac{\sqrt{1+\varkappa^2}}{\pi}\int_0^{\pi}\left(1-\frac{\varkappa_1}{2}\cos\alpha\right)d\alpha,$$

from which

$$R_{dyn\ mn}(\alpha) \approx R_{st}\sqrt{1+\varkappa^2}. \qquad (6.22)$$

2) $\kappa < 1$. Here the conventional moment of the frictional forces with values $\alpha = \alpha_\kappa$ changes its sign. According to (6.21) when $0 < \alpha < \alpha_\kappa$ the value of $\cos\varepsilon < 0$, $\varepsilon > \pi/2$ and consequently, the conventional moment of friction is directed toward the side opposite to the angular velocity ω. With $\alpha_\kappa < \alpha < \pi\cos\varepsilon > 0$, $\varepsilon < \pi/2$ and consequently, the direction of the moment coincides with the direction of the angular velocity.

In accordance with this, the effect of the action of the frictional forces may be evaluated using the following conventional mean value of the dynamic reaction modulus:

$$R_{dyn\ mn}(\alpha) \approx \frac{R_{st}\sqrt{1+\varkappa^2}}{\pi}\left[-\int_0^{\alpha_\varkappa}\left(1-\frac{\varkappa_1}{2}\cos\alpha\right)d\alpha+\int_{\alpha_\varkappa}^{\pi}\left(1-\frac{\varkappa_1}{2}\cos\alpha\right)d\alpha\right],$$

from which /214

$$R_{\text{dyn mn}}(\alpha) \approx R_{\text{st}} \frac{\sqrt{1+\varkappa^2}}{\pi} (\pi - 2\alpha_\varkappa + \varkappa_1 \sin \alpha_\varkappa). \tag{6.23}$$

It is not hard to see that, according to (6.21) when $\varkappa_1 \to 1, \alpha_\varkappa \to 0$, and therefore

$$R_{\text{dyn mn}}(\alpha) \to R_{\text{st}} \sqrt{1 + \varkappa^2}.$$

when $\varkappa \to 0$, $\alpha_\varkappa \to \pi/2$, $\varkappa_1 \to 0$, and consequently

$$R_{\text{dyn mn}}(\alpha) \to 0.$$

The examined instance of movement of a free, heavy pivot, still gives no idea of the actual effect of friction in the kinematic pairs on the behavior of a mechanism with elastic couplings, operating under the conditions of rotary vibration of the stand.

The results obtained are correct only if component 2 is loaded only by the force of weight. However, the components of mechanisms with elastic couplings are under the effect not only of constant or smoothly changing external forces, but also under the effect of forces of elastic couplings. As a rule, the system of forces acting on a mechanism with elastic couplings, and the limitations imposed on the relative motion of its components, are such that they exclude the possibility of rotation of the pivot of the component examined, with respect to the bearing. It is precisely then that slipping between the elements of kinematic pairs originates. In this case, the moment of frictional forces determined by us ceases to be conventional and, causing an additional deformation of the elastic couplings, becomes the cause of the withdrawal of the mechanism.

Returning to the example examined by us, we write the mean value of the moment of the frictional forces, which, according to (6.22), is equal to

$$M_{\text{mn}} = f r_{\text{p}} \sqrt{R_{\text{st}}^2 + M^2 \varrho_0^2 \omega^4}.$$

If the quasi-elastic coefficient of the system is equal to K, then its withdrawal, in the first approximation may be evaluated with the value

$$\Delta = f \frac{r_{\text{p}}}{K} \sqrt{R_{\text{st}}^2 + M^2 \varrho_0^2 \omega^4}. \tag{6.24}$$

It is not hard to see, that in this case, in contrast to the case examined in the preceding paragraph, the value of the withdrawal does not depend on the size of the clearance, and consequently, with a gap small enough to be disregarded, may achieve a substantial size depending on the intensity of vibration, rigidity of the elastic couplings, etc. /215

In the presence of not one but several kinematic pairs, friction moments causing the withdrawal of the mechanism, may arise only in those among them, by which the mechanism is attached to the vibrating stand. In this case, the friction moments, applied in different kinematic pairs, may cancel each other out. This situation is known and used in the instrument-designing practice.

In conclusion, we return to the instance examined by us, when the action of the elastic couplings is replaced with the action of a constant force (for example, the force of the weight of a component) and determine the law of motion of the pivot with respect to the bearing, keeping in mind that with the assumptions made by us, the pivot must roll about the bearing without slipping.

Here, its relative angular velocity Ω is connected with the angular velocity of rotation of vector $\overline{R}_{dyn}$ by the following ratio

$$\Omega = -\frac{r}{r_p}\frac{d\varphi}{dt}. \tag{6.25}$$

Function $\phi(\alpha)$ may be formed by projecting triangle O_2ab on the y axis (see Figure 6.6):

$$\cos\varphi = \frac{R_{st}(\varkappa\cos\alpha - 1)}{R_{dyn}}.$$

Substituting the value of R_{dyn} from (6.20), we get

$$\cos\varphi = \frac{\varkappa\cos\alpha - 1}{\sqrt{1+\varkappa^2-2\varkappa\cos\alpha}}.$$

Differentiating the latter equation we get for $\frac{d\phi}{d\alpha}$ the following expression:

$$\frac{d\varphi}{d\alpha} = \frac{\varkappa(\varkappa-\cos\alpha)}{1+\varkappa^2-2\varkappa\cos\alpha}. \tag{6.26}$$

Substituting (6.26) in (6.25), and keeping in mind that ω = const /216

$$\frac{d\varphi}{dt} = \frac{d\varphi}{d\alpha}\omega,$$

we get

$$\Omega = -\frac{r}{r_p}\frac{\varkappa(\varkappa-\cos\alpha)}{1+\varkappa^2-2\varkappa\cos\alpha}\omega.$$

The denominator of this equation represents a substantially positive value, and consequently, the sign of the angular velocity is determined by the sign of the numerator. It is not hard to see, that when $\kappa \geq 1$, the angular velocity of the pivot's rotation remains constant with respect to

direction. Here, the angular arrangement of the mechanism's component examined, with time, changes even with a small size of the clearance. The mechanism's withdrawal arises, the value of which may be determined in the following way:

$$\Delta = \int_0^t \Omega \, dt.$$

If we disregard the effect of the weight, or if the plane of the bearing coincides with the horizontal plane, then, keeping in mind that $\kappa = R_{in}/R_{st}$, we get

$$\Omega = -\frac{r}{r_p}\omega,$$

and consequently, the withdrawal will increase uniformly, while the component examined will turn in the direction, opposite to the direction of the angular velocity of the vibration.

With values $\kappa < 1$ the pivot will perform a periodic motion, returning with the lapse of time, to the initial position.

The analysis carried out in this chapter enabled us to make an evaluation of the effect of frictional forces in the kinematic pairs of a mechanism on its stability, dissipative properties, and dynamic precision.

As demonstrated, the existence of friction in the kinematic pairs does not exclude the possibility of origination of unstable and resonance systems of motion, the determination of which is connected with the evaluation of the dissipative properties of the mechanism. Friction in kinematic pairs does not /217 always exercise a stabilizing influence on the behavior of the mechanism under the conditions of established motion. In particular, friction in kinematic pairs, connecting the mobile components with the stand, along with the disbalance of the mechanism, and independently of it, may serve as the source of substantial dynamic errors of the mechanism.

However, the disbalance and friction do not exhaust the possible sources of dynamic errors of a mechanism, while the estimation of the forces of friction is not sufficient in the evaluation of its dissipative properties. A substantial influence both on the dynamic precision of the mechanism and on its dissipative properties is exercised by the presence of clearances in kinematic pairs.

The presence of clearances during the action of vibration or pulsation on a mechanism, results in the origination of special motion systems, accompanied by collisions in the kinematic pairs. Similar vibration-impact systems form also in the instances when the mechanism with elastic couplings has kinematic pairs with power locking, inasmuch as with intense vibration or pulsation the values of the preliminary tension, with which the elastic couplings are installed, proves insufficient, and does not provide for the closing of the kinematic system of the mechanism in individual time intervals.

Later, in Chapter 9, we shall return to the examination of the problems of dynamic precision of mechanisms with elastic couplings, and demonstrate there, the effect of collision in kinematic pairs on the values of dynamic errors.

CHAPTER 7

VIBRATION-IMPACT SYSTEMS

§ 7.1. Clearances in Kinematic Pairs. /218

Passing over to the examination of vibration-impact systems of motion of mechanisms with elastic couplings, we note first of all that for a number of machines and devices, such systems form the basis of their action. At the same time, frequently the vibration-impact interactions arise as a subsidiary phenomenon, accompanying the normal work of a mechanism. Such a substantial difference in the causes of origin of vibration-impact systems of motion does not prevent us, however, from examining with a single method the dynamics and stability of systems, in which these conditions are inherent.

In this chapter we shall examine a series of vibration-impact systems, build models, the investigation of which results from the analysis of these systems, and with a simple example, illustrate the methods of the analysis of the dynamics and stability of vibration-impact motion systems, which will be used in the subsequent chapters of the present book.

A substantial source of collisions in mechanisms are the clearances in their kinematic pairs. The existence of clearances, permitting the possibility of relative motion of the components, form the characteristic feature of any mechanism. The clearance in a kinematic pair is not simply the result of technological errors in the manufacture of its elements. The nominal sizes of clearances are determined by the designer on the basis of a number of considerations, proceeding from the functional purpose of the mechanism and the conditions of its operation.

The analysis of problems connected with the effect of the clearances on the accuracy of the mechanisms, their dynamics, and strength, had frequently become the subject of research by Soviet and foreign scientists. Here we may point out work [17], in which a number of problems in the field of mechanism /219
precision were formulated, in particular on the effect of gaps on the precision of mechanisms, as well as [111], in which the problems of estimation of the effect of gaps on the work of certain types of mechanisms are indicated.

As far as the author known, the first work in which the problems of dynamics of mechanisms are examined, with an allowance for the clearances, is [99]. However, extremely cumbersome graphic methods of analysis were used, and this work was not developed any further.

Certain general problems of accounting for the effect of clearances on the kinematic and dynamic precision of mechanisms are examined in [45]. These works are based on the assumption that the additional increments of generalized coordinates and their derivatives, originating as the result of

clearances, are small in comparison with the existing values, determined without an allowance for the clearances, and therefore, the effect of additional motion on the basic motion of the mechanism, may be disregarded. Later this approach was used applicably to mechanisms of various types in [64].

Problems of kinetostatic calculation of mechanisms with an allowance for the effect of clearances were developed somewhat in works [42, 61]. In them the problems of the effect of clearances on the distribution of forces arising in kinematic pairs of statically-indeterminable mechanisms are examined, as well as in the so-called asymmetrical mechanisms, the components of which move in parallel planes.

Certain problems of the dynamics of nonlinear sleeves with clearances are examined in [28] and others.

The problems of the movement of a pivot in a bearing with a clearance, important for various applications, were studied in [4], and then in [113]. In these works the authors established the relationships of the pivot motion system and the intensity of excitation, clearance size, and other factors. Important attention is given in the recent years to the theoretical and experimental investigations of the effect of clearances on the strength and service life of heavy machines [46].

This brief list does not exhaust the range of works in which, from one /220
point of view or another, an examination is made of the problems of the effect of clearances on the behavior of a mechanical system. A number of investigations will be mentioned later in connection with the examination of various vibration-impact systems. Here, we shall remark only that our review encompasses, incompletely, only those works in which the effect of the lubricating layer is not taken into account. The group of problems connected with the dynamics of systems, supported by the layer of lubricant, and the number of investigations devoted to their solution, are very extensive, however, the content of the above-mentioned problems differs substantially from the problems examined in the present book.

The presence of clearances changes substantially the conditions of the dynamic investigation problem. Disregarding their effect, we found it possible to limit ourselves in the preceding chapters to the examination of systems, possessing one degree of freedom. In the presence of clearances, the components of a mechanism acquire additional mobility. In particular, for example, the elements of the lowest pair, acquire in the relative movement two additional degrees of freedom. Consequently, the position of the mechanism in the presence of a clearance in one kinematic pair alone, in the general instance is characterized this time not only by one, but by three generalized coordinates.

In addition to the increase of the number of degrees of freedom, the presence of clearances introduces yet another feature both into the picture of the mechanism's motion, and into the problem of investigation of this motion. Disregarding the effect of clearances and examining the minor oscillations of the mechanism, we had obtained equations of motion in the form

of linear differential equations; the movement of the mechanism in this case had a "smooth" character.

In the presence of clearances, the motion of the mechanism even in the process of small oscillations, in the general case, is accompanied by "disruptions" of the kinematic chain and collision of the elements of kinematic pairs. Because of this circumstance, the problem of dynamic investigation becomes substantially nonlinear; linear equations are found inapplicable for describing motion as a whole, and remain valid only for its individual intervals in the times between the collisions.

We should mention, that here the motion of the mechanism has a certain characteristic feature, consisting of the fact that as a result of every /221
consecutive collision on its parts, linked with a clearance, their free oscillations originate. This feature of an established motion of a mechanism with a clearance, excludes the possibility of the use of the ordinary amplitude and phase characteristics for its dynamic analysis.

We had encountered such special established conditions of motion in other instances as well, for example in the study of the dynamics of cam mechanisms [37]. The profiles of the cams are usually composed of smoothly interconjugated sectors. Since in the point of conjugation, the radii of the curvature of two adjacent sectors, as a rule, are not equal to one another, the acceleration diagram of the tappet contains a "jump" in this point. With an established system of the work of a cam mechanism the acceleration jumps reoccur periodically, being the source of periodical excitation of free oscillations of the driven part of the system. We can mention a number of other mechanisms, the established operational systems of which are special, in the above-mentioned sense, and require for this evaluation methods which differ from the generally-accepted methods of amplitude and phase characteristics.

The two above-mentioned circumstances, the increase of the number of degrees of freedom of the system, and its nonlinearity, complicate the problem of the dynamic investigation appreciably, as the result of which, the rational way of solving it is the way of examining the dynamic models, which make it possible to determine in an illustrative way the specific features of the effect of clearances in kinematic pairs, which are the most important with one or another functional purpose of the mechanism.

§ 7.2. Dynamic Model of a Mechanism with a Clearance.

Before beginning to build such models it would be useful to clarify the question about the instances in which we may consider the size of the clearance small to disappearance and disregard its effect on the motion of the system, and in which instances this cannot be done.

For this purpose we shall turn to the case of the motion of a heavy balanced pivot under the conditions of the rotary vibration of the bearing (see Figure 6.6), examined in the preceding chapter. At that time we /222

disregarded the effect of the clearance and on this basis considered that the line of action of the dynamic reaction always passes through point A of the contact between the pivot and bearing, i.e., that

$$R_{in}\sin(\varphi-\alpha)=R_{st}\sin\varphi. \tag{7.1}$$

Let us show with which ratio between the parameters of the system and the actuation parameters this assumption remains correct, for which purpose we shall set up a differential equation of the pivot's movement with respect to the bearing, in the assumption of a uniform circular vibration of the latter.

The potential energy of the pivot with weight G is

$$\Pi=[\varrho_0\cos\alpha+r\cos(\alpha+\varphi)]\,G+\Pi_0.$$

The kinetic energy of the pivot is

$$T=\frac{J_{01}\Omega^2}{2}+\frac{MV_{01}^2}{2}.$$

Under the condition of rolling without slipping the absolute angular velocity of the pivot is

$$\Omega=\omega+\frac{r}{r-R}\dot{\varepsilon}.$$

The absolute velocity of the pivot's center of gravity is determined from the following relationship:

$$V_{01}^2=(\varrho_0\omega)^2+(\omega+\dot{\varepsilon})^2r^2+2\varrho_0r\omega\,(\omega+\dot{\varepsilon})\cos\varepsilon.$$

Having performed all the required computations, we shall obtain the differential equation of the pivot's motion in the following form:

$$\left[J_{01}\left(\frac{r}{r-R}\right)^2+Mr^2\right]\ddot{\varepsilon}+M\varrho_0r\omega^2\sin\varepsilon=Gr\sin(\alpha+\varepsilon).$$

Dividing both parts of the equation by $r\rho_0$, we get

$$\frac{r}{\varrho_0}\left[J_{01}\left(\frac{1}{r-R}\right)^2+M\right]\ddot{\varepsilon}+M\omega^2\sin\varepsilon=\frac{G}{\varrho_0}\sin(\alpha+\varepsilon).$$

It is not hard to see that equation (7.1) is fulfilled only under the condition that the r/ρ_0 ratio is small. In other words, we may disregard the influence of the clearance on the motion of a system only when its value is small in comparison with the amplitude of the bearing's motions, or in the general case in comparison with the sizes of the elements linked with the /223

clearance. The same conclusion may be drawn also in the presence of elastic couplings, which we can easily see by applying equation (4.34) to the case examined.

In the presence of a clearance two systems of motion of a mechanism with elastic couplings are possible in the process of its free or forced oscillations.

One of the systems is characterized by the fact that during all the period of the motion the elements of the pair having a clearance, are constantly in contact with one another (Figure 7.1). In other words, during the entire period of the motion, owing to the action of the elastic couplings, the force closure of the mechanism is assured, and for the full description of its motion it is sufficient to know two generalized coordinates (for example α and β in Figure 7.1,a).

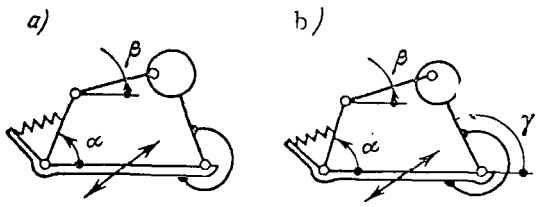

Figure 7.1.

The other possible system of motion is accompanied by alternation of moments of contact of the pair's elements with periods of their "free" relative motion within the bounds of the clearance. During the periods of such moments of the elements it is necessary to know the three generalized coordinates (for example, α, β, γ in Figure 7.1,b).

Both of the possible conditions differ substantially, with respect to the effect on the motion of the mechanism as a whole, and with respect to the methods which may be used for their investigation.

Investigating the small oscillations of the mechanism during its motion without disrupting the contact between the pair's elements, we shall arrive at a system of two differential equations, analogous to those which appeared in Chapter 4, i.e., at a system of linear equations with periodic coefficients.

The investigation of the second system of motion, accompanied by "breaks" /224 in the kinematic chain with subsequent collisions of individual parts of the mechanism, is incomparably more interesting. As the result of this investigation it becomes possible to evaluate the effect on the mechanism of such an extremely important dynamic factor as the action of impacts, periodically applied to the elements of the kinematic pairs.

The difficulties of such an investigation, as applied to the system in Figure 7.1, representing the general case of a mechanism with a clearance, were already mentioned in the preceding paragraph. Without distorting the essence of the problem, instead of this system we can examine another one, which simplifies the picture of motion accompanied with breaks and impacts.

Such a system of a mechanism with a clearance is given in Figure 7.2. For

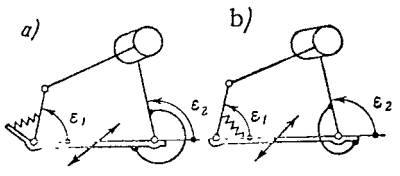

Figure 7.2.

it, just as for the system in Figure 7.1, two systems of motion are possible: without breaks and with breaks. However, in distinction from the system in Figure 7.1 the total number of degrees of mobility here is reduced to two. Consequently, in order to describe the motion of the mechanism, the kinematic chain of which is closed by one of the two methods presented in Figure 7.2,a or 7.2,b, it is sufficient to know one generalized coordinate. Upon a break of the kinematic chain, under which we shall understand the disruption of the contact between the cylindrical surfaces of the elements of the pair built with a clearance, the mechanism obtains a second degree of mobility.

In order to pass over to the analysis of the systems of motion, connected with breaks and impacts, and to building of the corresponding dynamic model, we shall remind the reader that in the case of the absence of clearances in kinematic pairs, the equation of small oscillations of the mechanism with elastic couplings during harmonic reciprocating vibration of the stand, had the appearance of (4.33)

$$\ddot{\varepsilon} + (\omega_{n\,\mathrm{dyn}}^2 + \varrho_0\omega^2\Psi_2\cos\omega t)\,\varepsilon = d + \varrho_0\omega^2\Theta_2\cos\omega t. \qquad (7.2)$$

Here ε is the generalized coordinate, read off from the position of the dynamic equilibrium α_{dyn}; the latter is characterized by the mean-integral value of the generalized coordinate, established in the process of forced oscillations of the mechanism /225

$$\alpha_{\mathrm{dyn}} = \lim_{t\to\infty}\frac{1}{t}\int_0^t \alpha\,dt,$$

where $\omega_{n\mathrm{dyn}}$ is the frequency of the free oscillations of the mechanism in the position corresponding to α_{dyn}, d is a constant value of translation, Ψ_2 and Θ_2 are periodicity and actuation coefficients, the values of which depend on the parameters of the mechanism and on its orientation with respect to the direction of the vibration. As it was shown, we can always select such values of the mechanism's parameters, or its position with respect to the direction of the vibration (see curves in Figure 5.1), with which $\Psi_2 = 0$. Then the coefficient with the generalized coordinate in equation (7.2) proves constant, and the equation of motion assumes the following appearance:

$$\ddot{\varepsilon} + \omega_n^2\varepsilon = \varrho_0\omega^2\Theta_2\cos\omega t. \qquad (7.3)$$

The stabilized motion conforming to this equation has a harmonic character and is performed near the position of static equilibrium. But this situation

occurs only in the case of the absence of a clearance in the kinematic pair, and in its presence only so long as the kinematic pair of the mechanism remains closed.

In order to clear up the picture of the motion of a mechanism in the presence of breaks, we shall set up two equations, describing the motions of both parts of the mechanism in the period between their collision. Having solved, thereupon, these equations jointly, we shall obtain the possibility of determining the amplitudes of free and forced oscillations of both parts of the mechanism, the middle positions, with respect to which they oscillate, the velocities of their collision, the amount of energy expended by the system in the process of collisions, etc.

We shall set up two equations describing the motion of individual parts of the mechanism, in the form of (7.3), i.e., disregarding the term containing coefficient Ψ_2.

Let us select as the generalized coordinates of the mechanism (see Figure 7.2) angles ε_1 and ε_2, reading them off from the position of static equilibrium. Then we shall get the following two equations, describing the motion of each of the parts of the mechanism in the periods between their collision: 7226

$$\left.\begin{aligned}\ddot{\varepsilon}_1 + \omega_{n_1}^2 \varepsilon_1 &= M_1 + \Theta_{12} Q_0 \omega^2 \cos(\omega t + \varphi),\\ \ddot{\varepsilon}_2 + \omega_{n_2}^2 \varepsilon_2 &= M_2 + \Theta_{22} Q_0 \omega^2 \cos(\omega t + \varphi).\end{aligned}\right\} \qquad (7.4)$$

In these equations ϕ is the angle characterizing the phase shift of the external excitation with respect to the moment adopted as the initial one, in the examination of the system's motion; ω_{n1}, ω_{n2}, Θ_{21}, Θ_{22}, represent respectively the free oscillation frequencies and actuation coefficients, determined for each of the parts of the system according to the rules indicated in Chapter 4; M_1 and M_2 are the generalized moments of the elastic couplings in the position of static equilibrium.

In Figure 7.3 we portray the models, dynamically equivalent to the systems in Figure 7.2.

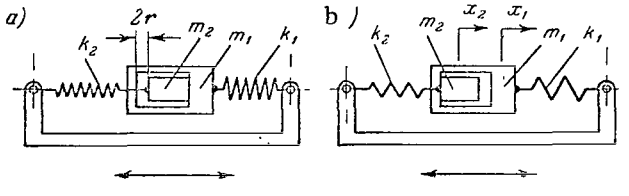

Figure 7.3.

In the intervals between collisions, the equation of motion of elements m_1 and m_2 of these models, with the harmonic vibration of the stand have the following appearances

$$\left.\begin{aligned}m_1\ddot{x}_1 + k_1 x_1 &= P_1 + m_1 Q_0 \omega^2 \cos(\omega t + \varphi),\\ m_2\ddot{x}_2 + k_2 x_2 &= P_2 + m_2 Q_0 \omega^2 \cos(\omega t + \varphi).\end{aligned}\right\} \qquad (7.5)$$

By comparing equations (7.4) and (7.5) we are convinced of their complete identity.

§ 7.3. Dynamic Models of Vibration-impact Systems. /227

As a typical example of a machine, for which the vibration-impact system is the basis of the technological process performed by it, we may use the spring hammers (see for example, [86]).

In Figure 7.4,a, we show the kinematic system of one of the designs of such a hammer. Here the power source rotates crank 1. From the crank, through connecting rod 2 and spring element k, the motion is transmitted to elements 3 and head 4 connected to them. Upon the rotation of the crank the head strikes the anvil, the frequency and intensity of the impact in this design being regulated by the number of revolutions of the crank. The initial regulation of the hammer's position is achieved by changing the length of the connecting arm. Apparently, under certain conditions, the motion of the hammer may be pretty close to periodical, even if we take into account the variations of the dimensions and properties of the blank being worked, which occur in the process of forging. It is also evident that such a periodical (or almost periodical) system of motion, with correctly-selected motion parameters, will be the most efficient and economical. From this point of view, the investigation of periodical stabilized systems of such machines and systems is extremely interesting.

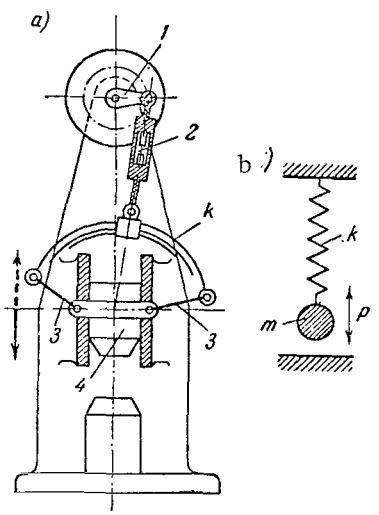

Figure 7.4.

There are spring hammers, the regulation of which, with a constant number of revolutions of the driving element is achieved through a change in the amplitude of the actuation applied to the elastic coupling. In both cases the investigation of the dynamics of spring hammers is reduced to the analysis of the dynamic model /228 shown in Figure 7.4,b. Here, m is the reduced mass of the forward-moving parts of the hammer, k is the rigidity of the elastic coupling, equivalent to the rigidity of the elastic element used in the hammer's design. As we can see, this model (as well as all the subsequent ones) is built in the assumption that the elastic elements possess linear characteristics. For a number of practically important instances, this assumption proves quite acceptable.

We arrive at the analysis of an analogous model through the investigation of the dynamics of the vibration-pneumatic tool, used widely in mining and building technology. In Figure 7.5 as an example, we give the kinematic diagram of one of the pneumatic hammer's designs. Here, crank 1 puts into reciprocating motion the slide bar 3. This motion is transmitted through

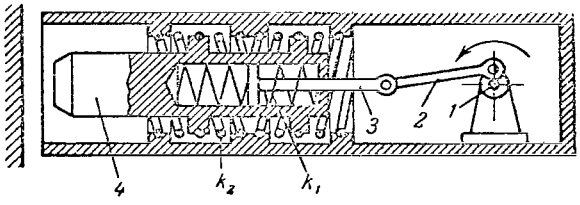

Figure 7.5.

elastic coupling k_1 to head 4, which is connected with the additional elastic coupling k_2 to the body of the hammer. Today we can mention several score vibration hammer designs, built according to this or similar systems. In both examples given above the forced motion of the driven element of the mechanism was actuated by means of a special mechanism through the periodic forced movement of the fixed end of the elastic element of the system.

Naturally, the periodic actuation in vibration-impact machines and devices may be achieved also by the use of vibrators.

In Figure 7.6,a, we give a simplified diagram of a pneumatic hammer, designed for driving piles and pile planks [79]. The pneumatic hammer consists of two parts: vibrating mass 1 with vibrators 2 and head 3, and the lower plate 4, connected rigidly with the pile or plank being driven. The vibrating mass is connected with the lower plate by two or four springs, so that, depending on the rigidity and initial adjustment of these springs the head and anvil may be pressed to one another or set up with a certain clearance. In both instances, upon the rotation of the disbalanced wheels directed oscillations of the vibrating mass may be created, accompanied by periodical collisions between the head and the anvil. /229

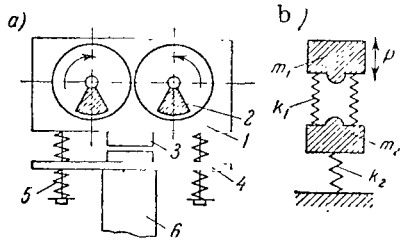

Figure 7.6.

It should be mentioned that the dynamic model portrayed in Figure 7.6,b, is suitable for investigating the vibration-driving processes only in the roughest approximation. Dynamic models which reflect the fact of the motion of the sinking pile and take into account the force of resistance to the motion of the pile are more exact (see [5, 79] and the literature quoted there). We shall not analyze such models here.

The theoretical investigation of the periodic cycles of the motion of a dynamic model, portrayed in Figure 7.4,b, was carried out initially in work [63] in connection with the development of the theory of a high-frequency generator of mechanical oscillations. Later, these results were used and developed in [54] and [103] in investigation of the dynamics of pneumatic tools, as well as in [5] and [79] applicably to the investigation of vibration-impact sinkers; in the second one of works [5] an examination is made also of the motion of the model given in Figure 7.6,b, which possesses two degrees of freedom.

In recent years, machines for vibration-impact testing of various instruments and special equipment [23] are becoming popular. The approximate kinetic models of these machines and their models are given in Figures 7.7 and 7.8.

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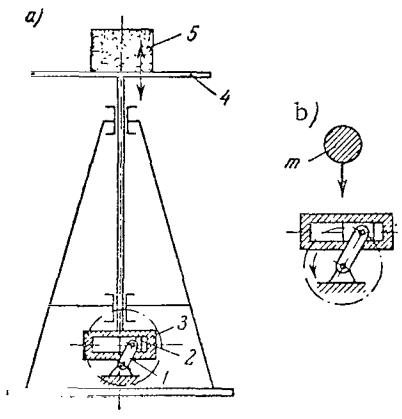

Figure 7.7.

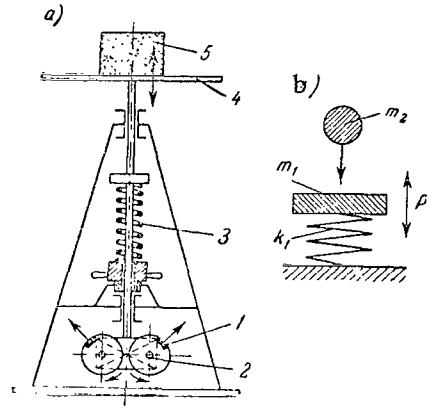

Figure 7.8.

In Figure 7. ,a, we portray the system in which the kinetic drive of the machine's table is used. Here, Table 4 with guide slot 3, slide block 2, and crank 1 form a sinusoidal mechanism. During uniform rotation of the crank the table moves harmonically. With a sufficient number of revolutions of the crank and its substantial length the object tested 5 will move, detaching itself periodically from the table and hitting it again. The corresponding dynamic model is given in Figure 7.7,b.

In Figure 7.8,a, we give the kinematic diagram of a machine for vibration-impact testing with power activation. Table 4 rests on an elastic element 3 and is connected with vibrator 2, carrying loads 1. The corresponding dynamic model is given in Figure 7.8,b. The difference between these two systems is obvious. In the first instance, i.e., with kinematic actuation of the table, the problem is reduced to the analysis of a system with one degree of freedom, inasmuch as the motion of the table is considered to be prescribed. In the second case, actuation of the table has a force character, and its motion, as well as the motion of the load may be found only as the result of investigation of the dynamics of a two-mass system. /231

To such dynamic models we come as the result of investigation of the dynamics of various machines and devices. As an example we shall mention certain vibration conveyers and knock-out grates which are widely used in the casting industry for knocking out the mold boxes, etc. We should also state that a substantial idealization is inherent in the dynamic models presented in Figures 7.7,b and 7.8,b, in comparison with the actual picture of the work of such machines. The circumstance that a small ball is adopted as the body falling on the vibrating base, reduces the collision problem to a direct central impact of a ball on a plane. Actually, the manufactured articles

undergoing tests for impact shocks (mold boxes interacting with knock-out grates, manufactured articles or materials transported on vibrating conveyers) may have the most diverse configuration, and then their position will have to to be determined not by one but by two or even three coordinates.

In Figure 7.9,a and b, we give the structural diagram and dynamic model of the knock-out grate of the original Stankolit Plant design. The mold box 1 to be knocked out is installed on supporting rails 2. The machine's table 3 equipped with a power actuator 4, in its upward motion strikes to mold box, lifting it off the support.

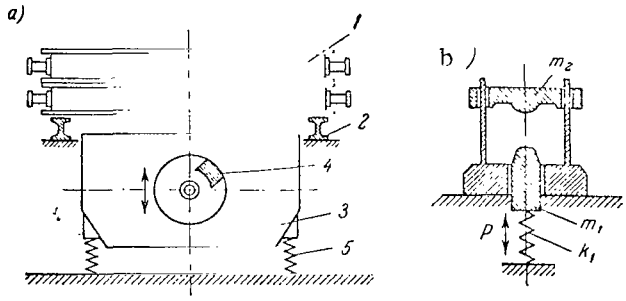

Figure 7.9.

In the reverse motion of the table the mold box sustains another shock in its encounter with the support. Thus, in a single period of the machine's motion, the mold box sustains two shocks, as the result of which the knocking-out efficiency is increased. /232

We shall not describe in greater detail the systems of the dynamic models of other types of vibration-impact machine action. The examples given provide us with a visual concept of the use of vibration-impact systems for realizing the wildest variety of technological processes.

The problems of the motion of bodies on vibrating surfaces as applied to the vibration-conveyer and vibration-separation processes were examined in works [12, 74] and others. The elements of the theory of machines acting according to the systems portrayed in Figures 7.7 and 7.8 are examined in [44, 47, 55, 56]. One of the first systems, intended specially for work under the vibration-impact operating conditions is an instrument invented in 1925 [2] and intended for measuring the amplitude of the telephone diaphragm oscillation (Figure 7.10,a).

The measurement principle in this device was based on the fact that with a certain frequency and amplitude of oscillations of diaphragm 1, "hammer" 2 begins to rattle, periodically bouncing off the diaphragm and hitting it again. The moment of origination of such a vibration-impact system is perceived easily by ear with the aid of microphone 3.

In Figure 7.10,b, we give the simplified dynamic model of this system, where m_1 is the mass of the diaphragm brought up to the center, k_1 is the rigidity equivalent to the diaphragm, m_2 is the mass of the hammer, and k_2 is the rigidity of the elastic element in which it is fixed. /233

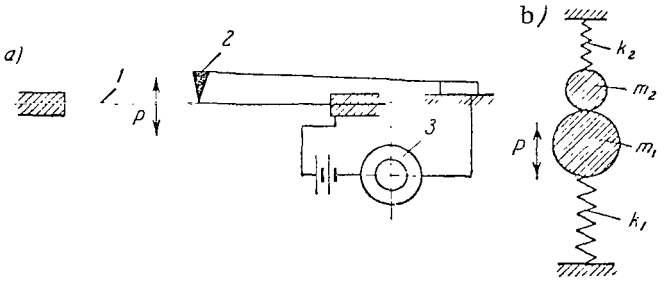

Figure 7.10

Today there is an appreciable number of perfected methods of measuring the oscillation parameters of vibrating bodies, however, the investigation of the motion of the dynamic model given in Figure 7.10,b, is extremely important for a number of other applications.

To a similar model, among others, comes the picture of the motion of a number of vibration-impact action machines, if we attempt to take into account the elasticity of the limiter; its analysis is the result of investigation of certain vibration-impact motion conditions of mechanisms with elastic couplings (Figure 7.3), and certain designs of impact-action vibration arresters (see below, Figure 7.14 and others).

The impact-action vibration arrester was invented in 1931 and used for eliminating [109] the vibration of turbine vanes.

Its design is given in Figure 7.11,a. The vibration-arresting element 1 represents a round rod, entering with a clearance an opening drilled in the body of vane 2. In the lower end of the rod a rectangular slot is cut, through which passes, with a large clearance, block 3, fixed in the vane's body. The centrifugal forces developed by the rod during the rotation of the rotor with the vanes are transmitted to the block through a steel ball 4. As a result, the friction value is reduced, and the rod is transformed into a three-dimensional pendulum, arresting the vibrations of the vane in any plane, passing through the rod's axis.

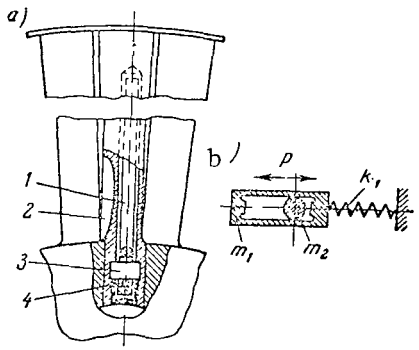

Figure 7.11.

The plane dynamic model of this vibration arrester is given in Figure 7.11,b. Here, m_1 and k_1 are the reduced mass and rigidity of the vane being damped, /234 m_2 is the mass of the vibration-arresting element. The damping effect in the use of such vibration dampers is reached both through the dynamic interaction of the principal system and the vibration-arresting element as the result of their collisions, and also through the dissipation of energy resulting from the fact that these collisions are not perfectly elastic.

In 1945 work [102] was published. The successful application of impact vibration arresters for eliminating vibrations in aircraft structures was mentioned in it (see also [1]). In Figure 7.12 we show a diagram of the position of the vibration arrester in an element of the aircraft's fins. A large number of vibration-damping elements provides the required effect along the entire length of the element. With respect to its other features, this structure is reflected sufficiently well by the dynamic model in Figure 7.11,b.

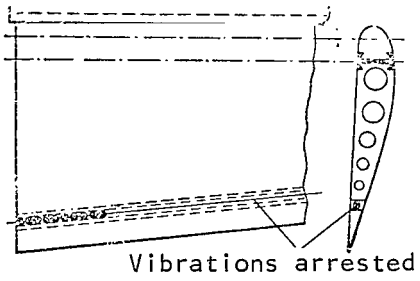

Figure 7.12.

In 1952 an original vibration arrester was invented [83], intended for eliminating high-frequency vibrations of cutters and boring bits, the design and dynamic model of which are given in Figure 7.13. Its structure is based on a massive bolt 1, which is screwed/235 into the holder or bit mandrel. Parts 2, 3 and 4 prevent the bolt from spontaneous loosening and protect the place where it is screwed in from penetration by dust and dirt. The required condition for the work of the vibration damper is the presence of a clearance in the threaded joint of the bolt and holder, through which their impact interaction is achieved in the process of vibration. The novelty of this invention consists of the possibility of regulation of the size of the clearance between the vibration-damping element and the system damped (see Figure 7.13), by means of changing of the length of the screwing together, inasmuch as the clearance size is one of the most important parameters, affecting the efficiency of the action of impact vibration dampers.

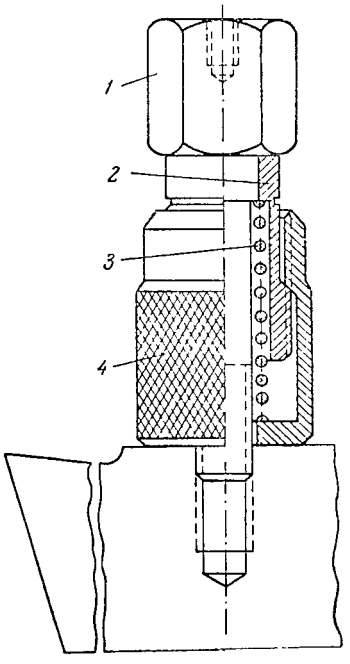

Figure 7.13.

Successful experiments in the use of this vibration damper brought about the appearance of other, simpler designs. One of them is given in Figure 7.14 (see [51]), its dynamic model corresponding to the model given in Figure 7.10. Application was found for designs in which the vibration arresting element is a ring set loosely on the vibrating machine part, [52], etc. The application of the method of impact vibration damping is not limited to instances when we want to eliminate high-frequency vibrations, occurring with small amplitudes, although it is precisely for these cases that it appears to be the most rational. Thus, for example, such a method was used for arresting the oscillations of entire tower-type structures [50].

Simultaneously with the ever increasing adoption of the impact vibration

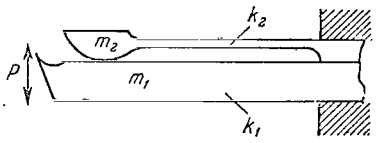

Figure 7.14.

damping, began the work on the development of its theory. Work [62] was one of the first. In it an examination was made of the motion of two pendulums, having a common swinging axis. The motion of the pendulums with respect to one another is limited by supports fixed on one of them. We should note that this system is dynamically equivalent to the design described in [50].

Various problems of the dynamics of impact-action vibration dampers with some simplifying assumptions, as well as the results of experimental investi- /236
gations are given in works [24, 43, 84, 85, 91]. The impact vibration damping theory was developed further in [39]. In it, along with the problems of dynamics, the problems of the stability of periodic working conditions of the motion of the elastic system, equipped with an impact vibration damper, are also examined by the method of finite differences. Simultaneously with [39], work [8] appeared, in which similar stability problems were solved by the point transformation method, as applied to the model portrayed in Figure 7.4,b.

In terminating the review of the works on the use and investigation of impact-action vibration arresters, we remark that the experience accumulated in this field enables us to pose more extensively the problem of the dissipative properties of mechanisms in connection with the impact interaction in their kinematic pairs. Up to the present time, the energy losses occurring in the process of the work of a mechanism are attributed only to the internal and external friction. At the same time, impact interactions are characteristic for the normal work of a number of mechanisms with lower pairs, gear-, cam-, and other mechanisms, while these impact interactions are unavoidably connected with dissipation of energy. In some instances, the share of this factor in the general balance of irretrievably scattered energy may prove considerable, and then the evaluation of the dissipative properties of a mechanism with clearances, along with the other problems of its dynamics, should represent a substantial interest.

In 1947, the authors of works [27 and 115], studying the effect of various nonlinearities on the dynamics of mechanical chains of control systems, simultaneously and independently of one another arrived at the examination of the dynamic model given in Figure 7.15,b, and imitating the clearance of any one of kinematic pairs, for example in a cogged transmission ("clearance" type nonlinear element). In work [27], in addition, an instance is examined when along with the clearance, an allowance is made for the elasticity of the driven system, as it is shown in Figure 7.15,c ("fork" type nonlinear element). An approximate evaluation of the dynamic properties of nonlinear elements of similar type was given in these works. The analysis carried out there is based on a series of simplifying assumptions, (see below in § 8.1), as the /237
result of which it was not possible to determine a number of characteristic features of the motion of the system with a clearance. Various aspects of the effect of clearances on the motion and stability of automatic control systems were investigated in a number of works (see, for example [22, 29, 105]).

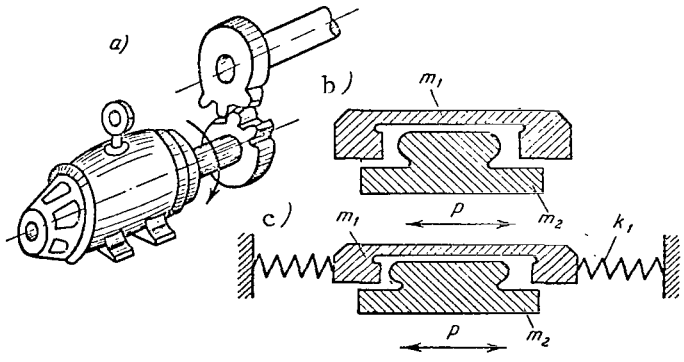

Figure 7.15.

However, in all these instances, the authors used the same simplifying assumptions, as the authors of [27] and [115].

Later, in works [19, 21] it was possible to abandon most of the simplifying assumptions upon examining the dynamics and stability of the nonlinear element with a clearance. In these works, as well as in [20] the analysis of the dynamic stability of vibration impact systems by the method of finite differences, was developed further. The stability of the periodic motions of a nonlinear element with a clearance by the method of point transformations was examined in [76].

We could extend substantially the review of the problems resulting in the investigation of vibration-impact systems. Thus, for example, investigations carried out by Ye. P. Novodvorskih and V.A. Shcherbakova indicated that the causes of dynamic errors of instruments operating under vibration conditions, may be the impact interaction of the elements of a kinematic chair, closed with elastic couplings (see also [39]). The corresponding dynamic model is given in Figure 7.16,a. Here, m_1 and m_2 are reduced masses of both parts of the kinematic chain of the instrument with force closure, which is attained through the initial tightness of the elastic couplings k_1 and k_2 (see the mechanism in 3.26, in which the kinematic pair formed by elements 2 and 3 is closed in precisely this way). This dynamic model differs from the one presented in Figure 7.3 by the unilateral (instead of bilateral) limitation of the mobility of both parts of the system. Speaking about different vibro-impact systems, we should also mention the large number of various mechanisms in automatic weapons (as an example, see Figure 7.16,b), relay-type mechanisms (see Figure 7.16,c), clock mechanisms, and others.

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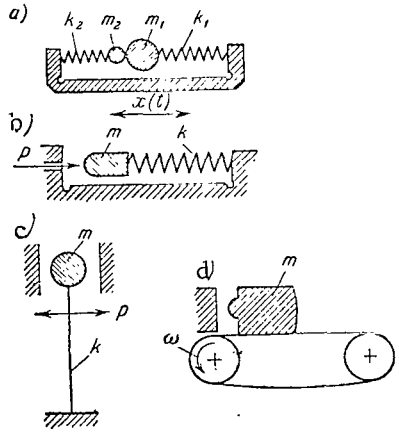

Figure 7.16.

In a number of cases the motion system, accompanied by collisions, has the character of natural oscillations. One of the possible models of such a natural-oscillation, vibration-impact system is portrayed in Figure 7.16,d; this model reminds us somewhat of the well-known [72] model, however, in distinction from it, the origination of natural oscillations here is not connected with the existence of the falling characteristic of friction between

the mass and the endless belt, moving at a constant speed. Natural-oscillation vibration-impact systems are formed in a number of cases by measuring devices, hydromechanical and pneumomechanical slave devices, having mechanical feedback circuits, etc.

As we can see, vibration-oscillation systems are used widely in many branches of technology. In this connection, as well as in connection with the fact that vibration-impact conditions often arise in various instruments and machines, as a side- but very unpleasant effect, the investigation of the /239 dynamics and stability of such conditions has a great practical value.

§ 7.4. Vibrator Jumping on Stairs.

In the subsequent chapters of this book we shall examine by the same methods some of the dynamic models given above, while at this time we shall learn about this method using the simplest example:

A ball jumping from step to step, here is probably the simplest vibration-impact system which we can imagine (Figure 7.17,a). We shall make this system somewhat more complicated by considering that, in addition to weight G*, the harmonic action $P_0^* \cos \omega t$ and the force of linear friction, equal to C*x* (C* > 0), are applied to the ball. (The asterisks with certain values here are used here and in the above to distinguish the dimensional from the dimensionless values in the future).

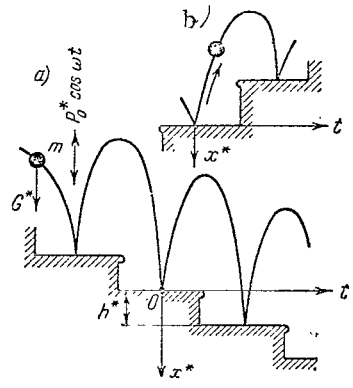

Figure 7.17.

The first problem of the investigation consists of establishing, under which conditions the motion of our elementary vibration will have a periodic character. Therefore, we shall proceed from the assumption that in the process of motion the vibrator does not miss any of the steps and does not hit any of them twice. We should state that posed in such a manner, the problem may have a specific practical interest. Thus, for example, we can model in the first approximation the work of certain designs of vibration pile drivers; the vibrator is equivalent to the impact system, and the staircase to a series of successive positions of the sinking pile.

We select the direction of the axes as it is shown in Figure 7.17,a. The time count will be made from the moment the vibrator hits the first one of the steps. Then, in the time interval before the impact against the next step, the motion of the vibrator will satisfy the following differential /240 equation:

$$m\ddot{x}^* + c^*\dot{x}^* = G^* + P_0^* \cos(\omega t + \varphi).$$

In this equation, angle ϕ characterizes the phase shift of the external force with respect to the moment of the start of the time count selected by us.

Turning to the dimensionless form of recording we get

$$\ddot{x} + c\dot{x} = 1 + P_0 \cos(\tau + \varphi), \tag{7.6}$$

where

$$x = \frac{\omega^2}{g} x^*, \quad c = \frac{c^*}{m\omega}, \quad P_0 = \frac{P_0^*}{G^*}, \quad \tau = \omega t, \quad \dot{x} = \frac{dx^*}{d\tau}.$$

Integrating equation (7.6) we find that dimensionless movements and the speed of the vibrator in the interval between two collisions are expressed in the following way:

$$\left.\begin{aligned} x &= C_1 e^{-c\tau} + C_2 + \frac{\tau}{c} + a\,[c \sin(\tau + \varphi) - \cos(\tau + \varphi)], \\ \dot{x} &= -C_1 c e^{-c\tau} + \frac{1}{c} + a\,[c \cos(\tau + \varphi) + \sin(\tau + \varphi)], \end{aligned}\right\} \tag{7.7}$$

where

$$a = \frac{P_0}{1 + c^2}.$$

Here C_1 and C_2 are integration constants which are usually determined in accordance with the prescribed initial motion conditions. However, we shall not employ this usual method of determining arbitrary constants, inasmuch as we cannot indicate ahead of time such initial conditions, with which the movement of the vibrator will have a periodic character. For determining C_1 and C_2, instead of the initial conditions we shall use the periodicity conditions, which are formulated in accordance with the motion conditions, the possibility of establishing of which we want to verify.

For example, we intend to find out with which ratios between the parameters of the vibrator and the parameters of actuation, such periodic motions may be established, that the period of impacts of the vibrators on the steps is equal to or a multiple of the period of the external force, while the speeds of impacts of the vibrator against all the steps are equal. Here, we shall assume that the time of the impact of the vibrator on the step is small in comparison with the period of its motion between the impacts, and this time may be not taken into account. The effect of the impact will be estimated by the restoration factor during the impact $0 \leq R \leq 1$. With the prerequisites indicated it is sufficient to examine only one interval of the vibrator's motion; the conditions of periodicity connect the positions and speeds of the vibrator at the boundaries of this interval, and are

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recorded as follows:

$$\left.\begin{aligned} &x = 0,\ \dot{x} = -Rx_0 \text{ when } \tau = 0,\\ &x = h,\ \dot{x} = x_0 \quad \text{when } \tau = 2\pi n \ (n = 1, 2, 3, \ldots), \end{aligned}\right\} \tag{7.8}$$

where x_0 is a dimensionless velocity of the vibrator's impact against the step, n is the multiplicity factor of the period of the vibrator's motion with respect to the external-force period.

Substituting the periodicity conditions (7.8) in expressions (7.7), we shall get the system of four equations, determining the arbitrary constants C_1 and C_2, as well as values x_0 and ϕ:

$$\left.\begin{aligned} 0 &= C_1 + C_2 + a\,[c\sin\varphi - \cos\varphi],\\ -R\dot{x}_0 &= -C_1 c + \frac{1}{c} + a\,[c\cos\varphi + \sin\varphi],\\ h &= C_1 e^{-2\pi nc} + C_2 + \frac{2\pi n}{c} + a\,[c\sin\varphi - \cos\varphi],\\ \dot{x}_0 &= -C_1 c e^{-2\pi nc} + \frac{1}{c} + a\,[c\cos\varphi + \sin\varphi]. \end{aligned}\right\} \tag{7.9}$$

Having solved these equations we find that

$$\left.\begin{aligned} &C_1 = \frac{2\pi n - hc}{c\,(1 - e^{-2\pi nc})},\quad C_2 = a\,[\cos\varphi + c\sin\varphi] - C_1,\\ &\dot{x}_0 = \frac{2\pi n - hc}{1 + R},\qquad \sin\varphi = \frac{-B \pm c\sqrt{aP_0 - B^2}}{P_0},\\ &B = -\frac{(2\pi n - hc)\,(1 + Re^{-2\pi nc})}{(1 + R)\,(1 - e^{-2\pi nc})} + \frac{1}{c}. \end{aligned}\right\} \tag{7.10}$$

Now all the values sought are expressed with the known parameters of the system and actuation. Equation (7.10) yields two different values of sin ϕ and accordingly, four values of the phase angle ϕ. The direct substitution of the values found into this equation indicates that two of the obtained values of ϕ satisfy all the conditions of periodicity. In other words, with the same values of c, R, h, and n, two different periodic-motion conditions may be present, to each of which corresponds its own value of the size of the impact phase, as it is shown, for example, by the solid and dotted lines in Figure 7.18. Moreover, with the same values of c, R, and h there may be periodical conditions of various multiplicity with respect to the frequency of the external periodical force. Thus, in Figure 7.19, we portray two different conditions of motion with n = 1 and n = 2 respectively. /242

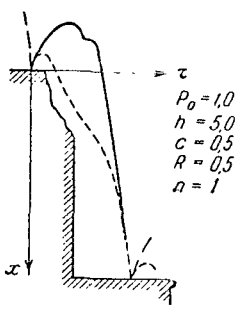

Figure 7.18.

This constitutes one of the substantial nonlinear peculiarities of the vibration-impact systems examined, in most of which the laws of periodical motion are multivalent.

The formula determining the value of impact ϕ phase (see (7.10)), enables us to determine the limits of the region of existence of periodical systems of motions of various multiplicity. Since sin ϕ should be an actual value, less than or equal to unity with respect to modulus, then /243

$$\left.\begin{aligned} |B| &\leqslant \frac{P_0}{\sqrt{1+c^2}}, \\ |\sin\varphi| &\leqslant 1. \end{aligned}\right\} \qquad (7.11)$$

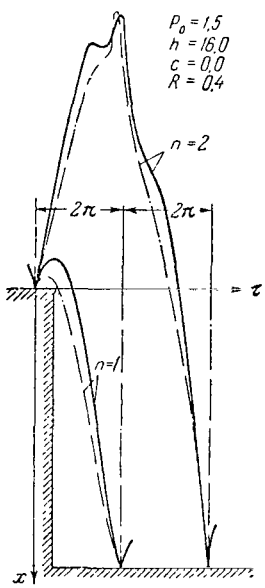

Figure 7.19.

The second one of the (7.11) inequalities proves stronger, inasmuch as it results in the following relationship between the parameters of the system and actuation:

$$|B| \leqslant \frac{P_0}{(1+c^2)}.$$

Using this expression we can find the relationship connecting the values of the parameters at the boundaries of the regions of existence. They will have the following appearance:

$$h_1 = \frac{2\pi n}{c} + \frac{(1+R)(1-e^{-2\pi nc})}{c^2(1+c^2)(1+Re^{-2\pi nc})}[P_0 c - (1+c^2)] \quad \text{for } B > 0,$$

$$h_1 = \frac{2\pi n}{c} - \frac{(1+R)(1-e^{-2\pi nc})}{c^2(1+c^2)(1+Re^{-2\pi nc})}[P_0 c + (1+c^2)] \quad \text{for } B < 0.$$

For the convenience of the further analysis we unite these two relationships, writing them as follows:

$$h_1 = \frac{2\pi n}{c} \pm \frac{(1+R)(1-e^{-2\pi nc})}{c^2(1+c^2)(1+Re^{-2\pi nc})}[P_0 c \mp (1+c^2)]. \qquad (7.12)$$

We note that the value of the linear friction coefficient is limited by the conditions of periodicity. Inasmuch as, according to (7.8) $\dot{x}_0 > 0$ always, from (7.10) we determine the following maximum value of coefficient c:

$$c_{\max} < \frac{2\pi n}{h}.$$

Later, examining the problems of stability of the periodic motions of the vibrator, we shall build the stability "charts" in axis P_0 and h. There, we shall have to return to equation (7.12). And at present we shall examine certain particular instances. /244

Let us assume that c = 0. Then, opening up the ambiguity found in this case, from (7.10) we get

$$\sin\varphi = \frac{2\pi^2 n^2 (1-R) - h(1+R)}{2\pi n P_0 (1+R)}. \tag{7.13}$$

Both values of the phase of impact ϕ, determined by this equation, satisfy the boundary conditions. As we can see, the absence of friction does not exclude multivalence. As before, several different periodic conditions are possible with the prescribed values of P_0, R, and h (see Figure 7.19).

The values of the regions of existence of periodic conditions with c = 0 are determined, according to (7.12), by the following equations:

$$h_1 = 2\pi^2 n^2 \frac{1-R}{1+R} \pm 2\pi n P_0. \tag{7.14}$$

The vibrator, under the action of the harmonic force may move both down (h > 0), and up (h < 0), as it is shown in Figure 7.17,b. The periodic motion on a smooth surface (h = 0) is maintained when

$$P_0 = \pi n \frac{1-R}{1+R} \tag{7.15}$$

and does not contain any specific features.

Substituting the value P_0 = 0 into (7.14) we shall arrive at a model representing a ball, freely jumping from step to step. As it follows from (7.15) on a smooth surface, such a free periodic motion of the ball is possible only when R = 1.

The case where R = 0 is quite interesting. In this case the speed of the ball after impact on the next step is fully attenuated. Depending on the ratios between the parameters, the dimensionless acceleration of the vibrator at the moment after the impact might be

$$\ddot{x}_0 \gtrless 0.$$

In the case where $\ddot{x}_0 > 0$ the vibrator after a certain time interval /245

remains pressed to the step, then the external force (under the condition that $P_0 > 1$) will "tear" it away from this step, and it will move until the moment of impact against the next step, etc. The periodic motion of the vibrator here will be added up from two stages and special conditions will be required in order to find the conditions of the existence of such systems. In the next chapter this instance will be examined in greater detail as applied to another dynamic model.

If, at the moment after the impact the dimensionless acceleration is found to be $\dot{x}_0 < 0$, the vibrator will be "torn away" from the step immediately after the impact; for such conditions the entire proceding analysis remains in force.

Evidently $\ddot{x}_0 = 0$ is a boundary case. We find the corresponding ratios between the parameters, doing this as applied to the case when $c = 0$. According to (7.6) at the moment after the impact ($\tau = 0$), $\ddot{x}_0 = 0$, if

$$P_0 \cos \phi = -1;$$

from which

$$\sin \varphi = \pm \frac{\sqrt{P_0^2 - 1}}{P_0}.$$

Comparing this expression for the impact phase with equation (7.13) when $R = 0$, we shall find that the maximum value of the dimensionless height of the step is

$$h_2 = 2\pi^2 n^2 \pm 2\pi n \sqrt{P_0^2 - 1}. \tag{7.16}$$

Thus, for $P_0 \geq 1$, provided $h \geq h_2$, the $R = 0$ case contains no peculiar features.

§ 7.5. Stability of Periodic Conditions of Motion.

The method of analysis of the stability of periodical vibro-impact conditions of motion which is used in this book is based on "fitting" of successive intervals of perturbed motion of the vibro-impact system examined. By means of fitting we may detect specific regularities in the changes of small perturbations, accumulating with the number of the fitted intervals, /246 tending toward infinity. If the small perturbation values will remain limited in this case, then the corresponding unperturbed motion will be considered stable; if we detect a tendency of their unlimited increase, then the corresponding unperturbed motion will be considered unstable.

Let us apply this method for the analysis of the stability of periodic

conditions of motion of our vibrator, jumping from step to step. We remind the reader that we sought the laws of its periodic motion in the form of equations (7.7). Let us assume that after one of the successive impacts the initial perturbation was introduced into this periodic motion, as the result of which the coordinate and speed of the vibrator became different from their calculated values. In equations (7.7) this initial perturbation will be reflected in the fact that constant integrations of C_1 and C_2 will obtain certain small increments, which we shall designate with ε_{10} and ε_{20}. The impact phase ϕ will also receive a corresponding small increment δ_0. As the result for the first interval of the perturbed motion of the vibrator, we get the following equation:

$$\left.\begin{aligned} x^{(1)} &= (C_1+\varepsilon_{10})\,e^{-c\tau}+(C_2+\varepsilon_{20})+\frac{\tau}{c}+ \\ &+a\,[c\sin(\tau+\varphi+\delta_0)-\cos(\tau+\varphi+\delta_0)], \\ \dot{x}^{(1)} &= -c\,(C_1+\varepsilon_{10})\,e^{-c\tau}+\frac{1}{c}+ \\ &+a\,[c\cos(\tau+\varphi+\delta_0)+\sin(\tau+\varphi+\delta_0)]. \end{aligned}\right\} \qquad (7.17)$$

In distinction from the unperturbed motion the perturbed motion is aperiodic, and the impact of the vibrator on the successive step will occur after time $\tau_1 = 2\pi n + \delta_1$, which differs from the calculated period by the small value δ_1. Substituting τ_1 in equation (7.17) and excluding the unperturbed motion, we shall find with a precision down to the small numbers of the second order, the increments of the dimensionless coordinate $\Delta x_E^{(1)}$ and speed $\Delta \dot{x}_E^{(1)}$ of the vibrator at the end of the first perturbed motion interval, expressed as functions of initial perturbations ε_{10}, ε_{20}, δ_0: /247

$$\left.\begin{aligned} \Delta x_E^{(1)} &= e^{-2\pi nc}\varepsilon_{10}+\varepsilon_{20}+\left(cC_1e^{-2\pi nc}+a\sin\varphi+\right. \\ &\left.+ac\cos\varphi+\frac{1}{c}\right)(\delta_0+\delta_1)+\left(cC_1e^{-2\pi nc}-\frac{1}{c}\right)\delta_0, \\ \Delta \dot{x}_E^{(1)} &= -ce^{-2\pi nc}\varepsilon_{10}+(c^2C_1e^{-2\pi nc}+a\cos\varphi- \\ &-ac\sin\varphi)(\delta_0+\delta_1)-c^2C_1e^{-2\pi nc}\delta_0. \end{aligned}\right\} \qquad (7.18)$$

Now let us examine the perturbed motion in the second interval. Here we shall read the dimensionless time τ from zero, combined with the moment of impact

$$\left.\begin{aligned} x^{(2)} &= (C_1+\varepsilon_{11})\,e^{-c\tau}+(C_2+\varepsilon_{21})+\frac{\tau}{c}+ \\ &\quad +a\,[c\sin(\tau+\varphi+\delta_0+\delta_1)-\cos(\tau+\varphi+\delta_0+\delta_1)], \\ \dot{x}^{(2)} &= -c\,(C_1+\varepsilon_{11})\,e^{-c\tau}+\frac{1}{c}+ \\ &\quad +a\,[c\cos(\tau+\varphi+\delta_0+\delta_1)+\sin(\tau+\varphi+\delta_0+\delta_1)]. \end{aligned}\right\} \qquad (7.19)$$

The motion of the vibrator in this interval will be fully determined if we know values ε_{11}, ε_{21} and δ_1, expressing the effect of the influence of the initial perturbation, introduced in the preceding interval.

This can be done by fitting both motion intervals on their common boundary. For this we substitute in (7.19) the value $\tau = 0$ and determine the increment of the dimensionless coordinate $\Delta x_B^{(2)}$ and speed $\Delta x_B^{(2)}$ at the beginning of the second interval of perturbed motion. In this case we shall get

$$\left.\begin{aligned} \Delta x_B^{(2)} &= \varepsilon_{11}+\varepsilon_{21}+a\,(c\cos\varphi+\sin\varphi)\,(\delta_0+\delta_1), \\ \Delta\dot{x}_B^{(2)} &= -c\varepsilon_{11}+a\,(\cos\varphi-c\sin\varphi)\,(\delta_0+\delta_1). \end{aligned}\right\} \qquad (7.20)$$

We shall consider that the heights of all the steps of the stairs on which the vibrator jumps, are equal. Then the fitting conditions should be written as follows:

$$\Delta x_B^{(2)} = \Delta x_E^{(1)} = 0; \qquad \Delta\dot{x}_B^{(2)} = -R\,\Delta\dot{x}_E^{(1)}. \qquad (7.21)$$

Here indices "B" and "E" indicate that the corresponding values pertain to the beginning or end of the motion interval, and the numerical indices indicate the number of the interval examined. /248

Substituting here the magnitudes of the values according to (7.18) and (7.20), we shall find a system of three equations, enabling us to find the sought values ε_{11}, ε_{21}, δ_1 as functions of the initial perturbation.

In this way, by substituting values $\tau_2 = 2\pi n + \phi_2$ into (7.19) we can find $\Delta x_E^{(2)}$ and $\Delta x_E^{(2)}$, and then write the perturbed motion equations for the third interval, and find $\Delta x_B^{(3)}$ and $\Delta x_B^{(3)}$. The successive fitting, analogous to (7.21) enables us to find ε_{12}, ε_{22} and δ_2 as functions of ε_{10}, ε_{20}, δ_0, etc.

Thus, we would be able to trace, one after another, a series of perturbed

motion intervals, and every time, determining values ε_{1i}, ε_{2i}, δ_i and comparing them with the values of the initial perturbations, approach the solution of the problem of the stability of one periodical condition or another.

Actually, however, there is no necessity for such countless passages from one motion interval to another. Using the method of finite differences, we can solve this problem much simpler.

Let us write the perturbed motion equation at a certain v-th interval. As before, we begin to count the dimensionless time from the moment of impact, and the effect of the entire preceding perturbed motion of the vibrator will be accounted for by perturbations $\varepsilon_{1(\nu-1)}$ and $\varepsilon_{2(\nu-1)}$, as well as by the accumulated perturbation of the phase of the v-th impact, which we shall write as $\Delta_{(\nu-1)} = \delta_0 + \delta_1 + \delta_2 + \ldots + \delta_{(\nu-1)}$. We shall consider values $\varepsilon_{1(\nu-1)}$, $\varepsilon_{2(\nu-1)}$, $\Delta_{(\nu-1)}$ to be small as before.

$$\left.\begin{aligned} x^{(\nu)} &= (C_1 + \varepsilon_{1(\nu-1)})\, e^{-c\tau} + (C_2 + \varepsilon_{2(\nu-1)}) + \frac{\tau}{c} + \\ &\quad + a\,[c \sin(\tau + \varphi + \Delta_{(\nu-1)}) - \cos(\tau + \varphi + \Delta_{(\nu-1)})]; \\ \dot{x}^{(\nu)} &= -c\,(C_1 + \varepsilon_{1(\nu-1)})\, e^{-c\tau} + \frac{1}{c} + \\ &\quad + a\,[c \cos(\tau + \varphi + \Delta_{(\nu-1)}) + \sin(\tau + \varphi + \Delta_{(\nu+1)})]. \end{aligned}\right\} \qquad (7.22)$$

Substituting $\tau_\nu = 2\pi n + \delta_\nu$, here we determine the increments $\Delta x_E^{(2)}$ and $\Delta x_E^{(2)}$ at the moment before the successive impact /249

$$\left.\begin{aligned} \Delta x_E^{(\nu)} &= \varepsilon_{1(\nu-1)}\, e^{-2\pi nc} + \varepsilon_{2(\nu-1)} - \left[C_1 c e^{-2\pi nc} - \frac{1}{c} - \right. \\ &\quad \left. - a\,(c \cos\varphi + \sin\varphi) \right] \Delta_\nu + \left(C_1 c e^{-2\pi nc} - \frac{1}{c} \right) \Delta_{(\nu-1)}, \\ \Delta \dot{x}_E^{(\nu)} &= -\varepsilon_{1(\nu-1)} c e^{-2\pi nc} + [C_1 c^2 e^{-2\pi nc} - \\ &\quad - a\,(c \sin\varphi - \cos\varphi)]\, \Delta_\nu - C_1 c^2 e^{-2\pi nc}\, \Delta_{(\nu-1)}. \end{aligned}\right\} \qquad (7.23)$$

In calculating these increments it was taken into account that

$$\delta_\nu + \Delta_\nu - \Delta_{(\nu-1)}.$$

Now, let us write the laws of the perturbed motion of the vibrator at the $(\nu + 1)$th interval

$$\left.\begin{aligned} x^{(\nu+1)} &= (C_1+\varepsilon_{1\nu})\,e^{-c\tau}+(C_2+\varepsilon_{2\nu})+\frac{\tau}{c}+ \\ &\quad +a\,[c\sin(\tau+\varphi+\Delta_\nu)-\cos(\tau+\varphi+\Delta_\nu)], \\ \dot{x}^{(\nu+1)} &= -c\,(C_1+\varepsilon_{1\nu})\,e^{-c\tau}+\frac{1}{c}+ \\ &\quad +a\,[c\cos(\tau+\varphi+\Delta_\nu)+\sin(\tau+\varphi+\Delta_\nu)]. \end{aligned}\right\} \tag{7.24}$$

Substituting value $\tau = 0$ here, we find the increments $\Delta x_B^{(\nu+1)}$ and $\Delta \dot{x}_B^{(\nu+1)}$ at the moment after the ν-th impact:

$$\left.\begin{aligned} \Delta x_B^{(\nu+1)} &= \varepsilon_{1\nu}+\varepsilon_{2\nu}+a\,(c\cos\varphi+\sin\varphi)\,\Delta_\nu, \\ \Delta \dot{x}_B^{(\nu+1)} &= -c\varepsilon_{1\nu}-a\,(c\sin\varphi-\cos\varphi)\,\Delta\nu. \end{aligned}\right\} \tag{7.25}$$

The conditions of fitting of two adjacent intervals yield

$$\left.\begin{aligned} \Delta x_B^{(\nu+1)} &= \Delta x_E^{(\nu)} = 0; \\ \Delta \dot{x}_B^{(\nu+1)} &= -R\,\Delta \dot{x}_E^{(\nu)}. \end{aligned}\right\} \tag{7.26}$$

Substituting here the magnitudes of the values according to (7.23) and (7.25), we shall get a system of three equations, connecting the values of perturbations determined through equidistant time intervals: /250

$$\left.\begin{aligned} &\varepsilon_{1\nu}+\varepsilon_{2\nu}+a\,(c\cos\varphi+\sin\varphi)\,\Delta_\nu=0, \\ &e^{-2\pi nc}\varepsilon_{1(\nu-1)}+\varepsilon_{2(\nu-1)}-\left[C_1ce^{-2\pi nc}-\frac{1}{c}-\right. \\ &\left.-a\,(c\cos\varphi+\sin\varphi)\right]\Delta_\nu+\left(C_1ce^{-2\pi nc}-\frac{1}{c}\right)\Delta_{(\nu-1)}=0, \\ &c\varepsilon_{1\nu}+Rce^{-2\pi nc}\varepsilon_{1(\nu-1)}+[a\,(1+R)\,(c\sin\varphi-\cos\varphi)- \\ &\qquad -C_1Rc^2e^{-2\pi nc}]\,\Delta_\nu+C_1Rc^2e^{-2\pi nc}\,\Delta_{(\nu-1)}=0. \end{aligned}\right\} \tag{7.27}$$

System (7.27) represents a system of linear homogeneous equations in finite differences, the solution of which is usually sought in the following form (see Chapter 2):

$$\left.\begin{aligned} \varepsilon_{1\nu} &= b_1\beta^\nu, \\ \varepsilon_{2\nu} &= b_2\beta^\nu, \\ \Delta_\nu &= b_3\beta^\nu. \end{aligned}\right\} \tag{7.28}$$

Here, coefficients b_i are independent of sign ν, β is the constant number to be determined.

Evidently, the solution of the problem of stability of the periodic motions of the vibrator is now reduced to the determination of the value of β. Indeed, if we find that $|\beta| < 1$, then with the number of impact $\nu \to \infty$ all the perturbation values will be

$$\varepsilon_{1\nu} \to 0, \quad \varepsilon_{2\nu} \to 0, \quad \Delta_\nu \to 0,$$

and consequently, the corresponding periodic motion will be stable.

In the opposite case, i.e., if we find that $|\beta| > 1$, then with $\nu \to \infty$

$$\varepsilon_{1\nu} \to \infty, \quad \varepsilon_{2\nu} \to \infty, \quad \Delta_\nu \to \infty,$$

the perturbed motion will deviate more and more from the unperturbed one, and the corresponding periodic motion will be unstable.

The case when $|\beta| = 1$ is a boundary case. Inasmuch as even small /251
variations of the parameters of the system and perturbation, which are always observed under actual conditions, may, in this case, result in an unlimited increase of perturbations, the unperturbed motion, corresponding to this instance, may be considered unstable.

In order to determine value β we shall act according to the rules mentioned in Chapter 2.

Let us substitute solutions (7.28) in equation (7.27). Then having reduced the first of these equations by β^ν, and two others by $\beta^{\nu-1}$, we shall get the following system for determining coefficients b_i:

$$\begin{aligned}
&b_1 + b_2 + a(c\cos\varphi + \sin\varphi)\, b_3 = 0,\\
&e^{-2\pi nc} b_1 + b_2 - \left\{\left[C_1 c e^{-2\pi nc} - \frac{1}{c} - a(c\cos\varphi + \sin\varphi)\right] + \right.\\
&\qquad\qquad \left. + \left(C_1 c e^{-2\pi nc} - \frac{1}{c}\right)\right\} b_3 = 0,\\
&c(\beta + Re^{-2\pi nc})\, b_1 + [a(1+R)(c\sin\varphi - \cos\varphi) -\\
&\qquad\qquad - C_1 R c^2 e^{-2\pi nc}]\, \beta b_2 + C_1 R c^2 e^{-2\pi nc} b_3 = 0.
\end{aligned}$$

For the existence of solutions different from zero, it is necessary that the determinant of this system would be equal to zero:

$$\begin{vmatrix} 1 & 1 & a(c\cos\varphi + \sin\varphi) \\ e^{-2\pi nc} & 1 & N \\ c(\beta + Re^{-2\pi nc}) & M & C_1 R c^2 e^{-2\pi nc} \end{vmatrix} = 0,$$

where

$$M = [a(1+R)(c\sin\varphi - \cos\varphi) - C_1 R c^2 e^{-2\pi nc}]\beta,$$
$$N = -\left\{\left[C_1 c e^{-2\pi nc} - \frac{1}{c} - a(c\cos\varphi + \sin\varphi)\right]\beta + C_1 c e^{-2\pi nc} - \frac{1}{c}\right\}.$$

Opening this determinant we shall have the following characteristic equation for determining the value of β:

$$d_0\beta^2 + d_1\beta + d_2 = 0, \tag{7.29}$$

where /252

$$d_0 = 1, \qquad d_2 = R^2 e^{-2\pi nc},$$
$$d_1 = -(1 + R^2 e^{-2\pi nc}) - \frac{(1+R)^2(1-e^{-2\pi nc})}{(2\pi n - hc)c} \times$$
$$\times \left[P_0\cos\varphi + \frac{cR(2\pi n - hc)}{1+R} - \frac{c(2\pi n - hc)}{1-e^{-2\pi nc}} + 1\right].$$

The analysis of the characteristic equation enables us to separate the regions of the values of parameters of the system and perturbation, corresponding to the stable and unstable motion systems.

§ 7.6. Stability Charts.

The unperturbed motion of the vibrator is stable if the roots of the characteristic equation (7.29) satisfy the following inequalities

$$|\beta_{1,2}| < 1 \tag{7.30}$$

with the parameter values corresponding to the periodic conditions examined.

To determine the limits of the regions of stable conditions we shall use Schur's theorem (see § 2.10). In this case the following criteria, assuring the fulfillment of the (7.30) conditions, are present:

$$\left|\frac{d_2}{d_0}\right| < 1, \qquad \left|\frac{d_1}{d_0 + d_2}\right| < 1.$$

The first of these inequalities after the substitution of coefficients d_2 and d_0 assumes the appearance of

$$|R^2 e^{-2\pi nc}| < 1$$

and is fulfilled with any values of the parameters, inasmuch as for a real physical system we always have R < 1 and coefficient c $\geq$ 0.

The second inequality will be written as follows:

$$\left|1+R^2e^{-2\pi nc}+\frac{(1+R)^2(1-e^{-2\pi nc})(P_0\cos\varphi+cB)}{(2\pi n-hc)c}\right|<1+R^2e^{-2\pi nc}. \tag{7.31}$$

It is convenient to plot stability charts of periodic motions of a vibrator on axes (P_0, h); in this case the lines separating the regions of stable and unstable conditions will represent functions $h = h(P_0)$, calculated with fixed values of parameters R, c and n. /253

The last inequality enables us to determine two limits in each one of the regions of stable periodic conditions for values n = 1, 2, 3, ... etc. Let us find one of them using the following ratio proceeding from this inequality:

$$\frac{(1+R)^2(1-e^{-2\pi nc})(P_0\cos\varphi+cB)}{(2\pi n-hc)c}<0.$$

Inasmuch as $2\pi n/h > c > 0$, it means that this inequality is fulfilled only under the condition that

$$P_0\cos\varphi+cB<0. \tag{7.32}$$

And now we return to equations (7.9), obtained in § 7.4. The last one of them, after substitution of values $\dot{x}_0$ and C_1 in it, results in the following equation:

$$a[\sin\varphi+c\cos\varphi]+B=0,$$

or

$$P_0\cos\varphi+cB=-\frac{P_0}{c}\sin\varphi-\frac{B}{c}. \tag{7.33}$$

Substituting here the value of sin ϕ, according to (7.10) we find

$$P_0\cos\varphi+cB=\mp\sqrt{aP_0-B^2}.$$

Turning to inequality (7.32) we now find that only those motion conditions are stable, which correspond to the plus sign under the root in formula (7.10), determining the impact phase. We shall find the limit of the stability region from the maximum equality corresponding to inequality

(7.32)

$$P_0 \cos \varphi + cB = 0.$$

According to (7.33), in this case

$$\sin \varphi = -\frac{P_0}{B}.$$

Comparing this equality with (7.10) we shall find that the limit found coincides with the limit of the region of actual values of sin ϕ, in other words, with the previously determined limit of h_1 of the region of existence of periodic conditions (see (7.12)). /254

We shall find the second limit of each of the regions of stability, using the inequality

$$\frac{(1+R)^2(1-e^{-2\pi nc})(P_0 \cos \varphi + cB)}{(2\pi n - ch)c} > -2(1+R^2e^{-2\pi nc}).$$

proceeding from (7.31). The corresponding relationship will have the following appearance:

$$P_0 \cos \varphi + cB = -\frac{2c(2\pi n - ch)(1+R^2e^{-2\pi nc})}{(1+R)^2(1-e^{-2\pi nc})}.$$

Using equations (7.33) and (7.10) we find, in the final count, the following equation for the second limit of each instability region:

$$h_3 = \frac{2\pi n}{c} - \frac{(1+R)^3}{c^2 D}(1-e^{-2\pi nc})(1+Re^{-2\pi nc}) \pm$$
$$\pm \frac{(1+R)^2}{cD}(1-e^{-2\pi nc})\sqrt{aD - 4(1+R^2e^{-2\pi nc})^2}, \qquad (7.34)$$
$$D = 4c^2(1+R^2e^{-2\pi nc})^2 + (1+R)^2(1+Re^{-2\pi nc})^2.$$

Before beginning to discuss the results of the analysis of the stability of periodic motion conditions of the vibrator, we shall indicate that, if we do not take friction into account (c = 0), equation (7.34) will assume the following appearance:

$$h_3 = 2\pi^2 n^2 \frac{1-R}{1+R} \pm 4\pi n \frac{1+R^2}{(1+R)^2}\sqrt{\frac{(1+R)^4 P_0^2}{4(1+R^2)^2} - 1}. \qquad (7.35)$$

The results of the analysis performed are given in Figures 7.20, 7.21 and 7.22 in the form of three stability charts. In Figure 7.20 we bring out the regions of stability of the multiples of periodic conditions (n = 1, 2, 3).

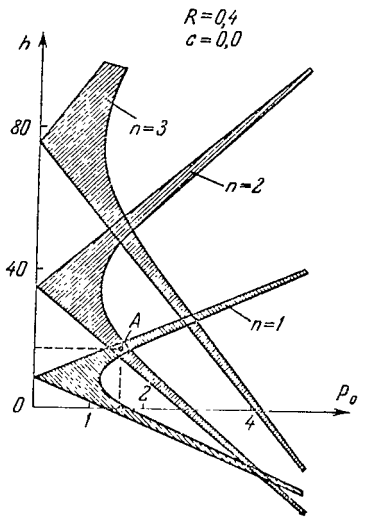

Figure 7.20.

As we can see, with the same values of R and c, the periodic conditions of motion of various multiplicity (for example n = 1 and n = 2, n = 1 and n = 3, etc) prove possible and stable. To point A on this chart correspond the laws of motion, presented in Figure 7.19 on which the stable conditions are plotted with solid lines. Which of these two stable conditions is actually realized by the system, will probably depend on the initial conditions and character of the transition motion of the vibrator. Inasmuch as we did not look into this "prehistory" of the periodic motion, in the general instance (i.e., with not specially selected initial conditions), we cannot give an exhaustive answer to this problem. /255

The questions: what will be the dynamic picture with other combinations of parameters of the system and actuation, to what periodical and non-periodical conditions of motion correspond the regions remained unshaded on the stability chart? also remain open.

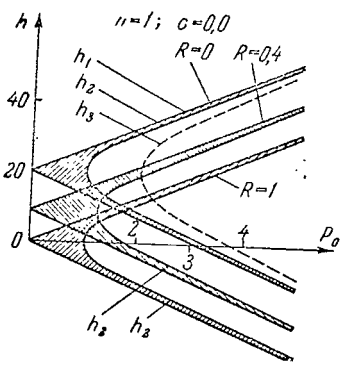

Figure 7.21.

Finally, we should note that the linearized equations of perturbed motion yield no precise answer to the question on the stability of conditions, taking place with parameter values, corresponding to the limits of stability regions, and in particular, give no answer to the question about the stability of the periodic conditions of the free motion of the ball jumping on the steps.

Thus, the simple example examined by us, enabled us not only to demonstrate the methods of analysis of the dynamics and stability used later, but also helped us bring out certain features, peculiar to nonlinear vibration impact systems and those limitations of the analysis results, which are the result of these features.

In Figure 7.21 we show the stability chart plotted for various values of the coefficient of restoration during impact. We note, first of all that the R = 1 instance which we have not examined separately in the analysis of stability, contains no specific features, with the one exception, that the /256

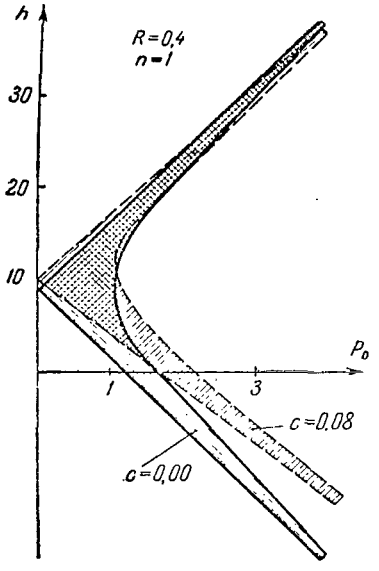

Figure 7.22.

corresponding region on the stability chart is located symmetrically with respect to axis P_0. For the R = 0 value the region of stable conditions within, is limited by line h_2 (see equation (7.16)), therefore line h_3 is drawn as a dotted line.

Finally, in Figure 7.22 it is shown that the presence of linear friction has little effect on the position and area of the stability region of periodic conditions, of course, under the condition that the friction coefficient c is not very large.

In conclusion we remark, that all the vibro-impact systems enumerated in this chapter may be divided into two groups.

For some systems it is characteristic that their motion is accompanied by unilateral collisions. Thus, in particular, move the above-mentioned vibrator, jumping down the stairs, the pendulum, hitting the immobile limiter (Figure 1.4), the jackhammer (Figure 7.4), and a number of others. In other instances (see for example, Figures 7.12, 7.15) the relative motion of both parts of the system has a bilateral limitation and is accompanied by collisions with first one and then the other limiting surface. If the restitution coefficient during the collision of both surfaces is equal, the periodic movement of such systems proves, to a certain extent, symmetrical and such systems will be called symmetrical; we shall consider the vibro-impact systems, the motion of which is accompanied by unilateral collisions, asymmetrical. /257

CHAPTER 8

SYMMETRICAL VIBRO-IMPACT SYSTEMS

§ 8.1. Nonlinear Element of the "Clearance" Type. /258

We shall begin the analysis of the dynamics and stability of symmetrical vibro-impact systems with the examination of the simplest model shown in Figure 7.15,b. There it was already mentioned, that we are led to the examination of this model by the problem of the effect of clearances on the dynamics of the automatic control systems, and two works [27] and [115] were mentioned, in which the problems of determination of the dynamic characteristics of nonlinear elements, included in the control system, and in particular, the nonlinear element with a clearance, were formulated for the first time. The analytical and graphoanalytical methods of solution developed in these works are approximate, and had been used with various assumptions with respect to the most important properties of the nonlinear element with a clearance.

Thus, in work [27] it is considered that mass m_2 of the driven component (Figure 8.1) is negligibly small and that the driven component moves only during periods of its contact with the driving component; it stops instantaneously as soon as this contact is disrupted. The connection between the movements of both components is visually illustrated by the so-called clearance characteristic, shown in Figure 8.2. In Figure 8.3,a, we give the corresponding law of motion of a system with a clearance with a harmonic external force acting on it. /259

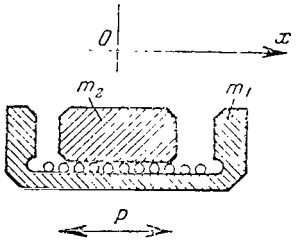

Figure 8.1.

In work [115] the masses of both parts of the system coupled with a clearance are taken into account. However, its motion with periodic collisions is examined with the assumption that the velocity restitution coefficient during collisions is equal to zero and both masses after the collision, for a certain time, are moving jointly, as it is shown in Figure 8.3,b.

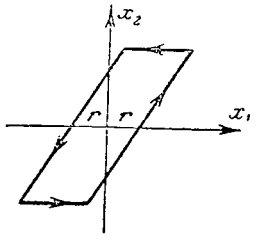

Figure 8.2.

Evidently, in a number of instances, certain assumptions adopted in the works mentioned, may prove quite acceptable. At the same time, the analysis limited by such prerequisites, does not permit us to uncover and explain a number of features peculiar to the motion of the system containing the clearance. For this, it is necessary at the same time to take into account at least two extremely important properties of the system, connected with the inertia of the individual parts, coupled with a

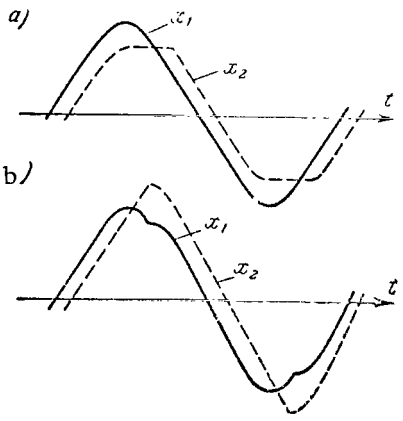

Figure 8.3.

clearance, and with their elasticity which manifests itself in the process of impact interaction. The first paragraphs of the present chapter will be on the subject of the solution of this problem.

As the basis of the solution we use the assumption that upon the action of a periodical force on a system, periodic motions may be established the period of which is equal to or a multiple of the period of the external force. To find these solutions we will, as before, use the periodicity conditions. Just as in the example with the vibrator jumping down the stairs, we shall consider that the time of collision is small in comparison with the period of motion, and therefore is not taken into account, while the entire collision process is evaluated by the coefficient of restitution upon the impact, which may have any values $0 \leq R \leq 1$, which, however, are equal for the collision of the right and collision of the left planes, limiting the relative motion of both parts. These prerequisites will also remain in force in the examination of all the other vibro-impact systems, and in the future we shall not specify them. /260

And so, let one of the parts of the system (m_1 in Figure 8.4) be under the load of a harmonic force $P = P_0 \cos \omega t$. Both masses are coupled with a clearance the full size of which is equal to 2r. Under the action of force P both masses begin to move, the source of the motion of mass m_2 being its impact interaction with mass m_1. Moreover, let mass m_2 be under the action of the force of dry friction F*.

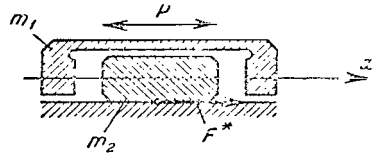

Figure 8.4.

We direct axis x as it is shown in Figure 8.4. Then, in the interval between collisions the motion of the system under examination is described by the following differential equations

$$m_1\ddot{x}_1^* = P_0 \cos(\omega t + \varphi); \qquad m_2\ddot{x}_2^* = -F^* \operatorname{sign} \dot{x}_2. \tag{8.1}$$

$$\operatorname{sign} \dot{x}_2 = 1, \quad \text{if} \quad \dot{x}_2 > 0,$$

$$\operatorname{sign} \dot{x}_2 = -1, \quad \text{if} \quad \dot{x}_2 < 0,$$

where ϕ is the phase shift of the external force with respect to the moment of the beginning of the time count.

To simplify further computations we shall introduce the dimensionless values: $\tau = \omega t$; $a = P_0/\omega^2 m_1$; $F = F^*/P_0$; $\mu = m_1/m_2$; $F^*/\omega^2 m_2 = \mu a F$; $\sigma = r/a$; $x_1 = x_1^*/a$; $x_2 = x_2^*/a$. Then the equations of motion (8.1) will be written thus:

$$\ddot{x}_1 = \cos(\tau + \varphi); \qquad \ddot{x}_2 = -\mu F \operatorname{sign} \dot{x}_2. \tag{8.2}$$

We shall find periodic motions characterized by the alternation of collisions of right and left planes of masses m_1 and m_2 under the condition that in $2n + 1$ ($n = 0, 1, 2, \ldots$) periods of the external force, a total of two collisions occurs. /261

The motion of the system may have a conditional periodic character if the coordinates and velocities of the system at the moments of collision of right and left planes will be equal in value and opposite in sign. Because of this symmetry the full period of motion of the system will be divided into two entirely identical intervals, encompassing the following half-periods: $0 - \pi(2n + 1)$ and $\pi(2n + 1) - 2\pi(2n + 1)$.

We shall refer the zero to the moment after the collision of the right planes (see Figure 8.4). Then the motion periodicity conditions for the first half-period will have the following appearance

$$\left.\begin{aligned} &x_1 = x_c; \quad x_2 = x_c + \sigma; \quad \dot{x}_1 = v_2; \quad \dot{x}_2 = \lambda_2 \\ &\text{with } \tau = 0, \\ &x_1 = -x_c; \quad x_2 = -(x_c + \sigma); \quad \dot{x}_1 = v_1; \quad \dot{x}_2 = \lambda_1 \\ &\text{with } \tau = \pi(2n + 1), \end{aligned}\right\} \tag{8.3}$$

where x_c is the collision coordinate, v_2 and λ_2 are dimensionless velocities of masses m_1 and m_2 respectively, after the collision of the right planes, v_1 and λ_1 are dimensionless velocities of these masses before the collision of the right planes, σ is the relative clearance.

In a similar manner we can set up the conditions of periodicity for the second half-period, by reversing all the signs in (8.3). Here we shall get

$$\left.\begin{aligned} &x_1 = -x_c; \; x_2 = -(x_c + \sigma); \; \dot{x}_1 = -v_2; \; \dot{x}_2 = -\lambda_2 \\ &\text{with } \tau = \pi(2n + 1), \\ &x_1 = x_c; \quad x_2 = x_c + \sigma; \quad \dot{x}_1 = -v_1; \; \dot{x}_2 = -\lambda_1 \\ &\text{with } \tau = 2\pi(2n + 1). \end{aligned}\right\} \tag{8.4}$$

Both half periods of motion are connected on their common boundary by the pulse theorem (see equation (1.11)), according to which /262

$$\left.\begin{aligned} -v_2 &= \frac{(1+R)\,\lambda_1+(\mu-R)\,v_1}{1+\mu}; \\ -\lambda_2 &= \frac{(1-\mu R)\,\lambda_1+\mu\,(1+R)\,v_1}{1+\mu}. \end{aligned}\right\} \tag{8.5}$$

Now, integrating equations (8.2), we find the laws of motion of both parts of the systems in the interval between collisions. Here we should take into account that during each half period mass m_2 under the effect of the constant force F moves with equal deceleration, so that its velocity $\dot{x}_2$, while it decreases in value, at the same time remains equal in sign. This circumstance simplifies the problem of integration, inasmuch as here, to each half-period of motion corresponds one definite sign in the second equation (8.2). Thus, for the first half-period $\dot{x}_2$ is always smaller than 0, and therefore, the equations of motion for this half-period will be

$$\ddot{x}_1 = \cos(\tau+\varphi); \qquad \ddot{x}_2 = \mu F. \tag{8.6}$$

The general solution of this system of equations has the following appearance

$$\left.\begin{aligned} x_1 &= -\cos(\tau+\varphi)+C_1\tau+C_2, \\ x_2 &= \frac{\mu F\tau^2}{2}+C_3\tau+C_4. \end{aligned}\right\} \tag{8.7}$$

Using this solution, the periodicity conditions (8.3), and the pulse theorem, we determine the unknown values of C_1, C_2, C_3, C_4, ϕ, x_c, λ_1, λ_2, v_1, v_2 so that the motion of the system would have a conditioned periodic character. As the result we shall get

$$\begin{aligned} C_1 &= -(2n+1)\,\frac{\pi F}{2}-\frac{\lambda_2}{\mu}, \\ C_2 &= (2n+1)^2\,\frac{\pi^2 F}{4}+(2n+1)\,\frac{\pi\lambda_2}{2\mu}, \\ C_3 &= \lambda_2, \\ C_4 &= -(2n+1)^2\,\frac{\pi^2\mu F}{4}-(2n+1)\,\frac{\pi\lambda_2}{2}, \end{aligned}$$

and

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$$\begin{aligned} x_c &= -\left[(2n+1)^2\,\frac{\pi^2\mu F}{4}+\sigma+(2n+1)\,\frac{\pi\lambda_2}{2}\right], \\ \lambda_1 &= (2n+1)\pi F\mu+\lambda_2, \\ v_1 &= -(2n+1)\pi F\,\frac{1-\mu R}{1+R}+\frac{\mu\,(R-1)-2}{\mu\,(1+R)}\lambda_2, \\ v_2 &= -(2n+1)\pi F\,\frac{R\,(1+\mu)}{1+R}+\frac{(1-R)\,\mu-2R}{\mu\,(1+R)}\lambda_2. \end{aligned}$$

The value of ϕ is determined by two relationships

$$\sin\varphi = -(2n+1)\pi F\frac{2\mu R+R-1}{2(1+R)}+\frac{(1+\mu)(1-R)}{\mu(1+R)}\lambda_2,$$
$$\cos\varphi = \frac{(2n+1)^2\pi^2}{4}F(1+\mu)+\sigma+(2n+1)\pi\frac{1+\mu}{2\mu}\lambda_2.$$

As we can see, all the values sought are now expressed in the form of linear functions of value λ_2. In its turn, it is determined by the following formula

$$\lambda_2 = -\frac{\mu}{(1+\mu)\left[(2n+1)^2\frac{\pi^2}{4}+\left(\frac{R-1}{R+1}\right)^2\right]}\left\{(2n+1)\frac{\pi}{2}\times\right.$$
$$\times\left[\sigma+F\left(\frac{(R-1)(2\mu R+R-1)}{(1+R)^2}-(2n+1)^2\frac{\pi^2}{4}(1+\mu)\right)\right]\mp$$
$$\left.\mp\sqrt{(2n+1)^2\frac{\pi^2}{4}+\left(\frac{R-1}{R+1}\right)^2-\left[\frac{R-1}{R+1}\sigma-(2n+1)^2\frac{\pi^2}{4}F\mu\right]^2}\right\}. \qquad (8.8)$$

Here we again encounter the already familiar property of ambiguity of periodic conditions of motion of vibro-impact systems. With the same parameters (μ, σ, R) of the system, and (a, ω) of the actuation, 2(n + 1) various values of λ_2 are possible and accordingly 2(n + 1) various periodic motion conditions.

The region of the actual values of λ_2 is determined from the equation

$$0 \leqslant \sigma \leqslant \frac{1+R}{1-R}\left[\sqrt{(2n+1)^2\frac{\pi^2}{4}+\left(\frac{1-R}{1+R}\right)^2}-(2n+1)^2\frac{\pi^2}{4}F\mu\right]. \qquad (8.9)$$

In Figure 8.5, as an example, we give the characteristics $\lambda_2 = \lambda_2(\sigma)$, plotted for n = 0 (two collisions in one period of the action of the perturbing force) and for various values of the dimensionless force of friction F. The dash line separates the region of values λ_2 within the limits of which the condition of constancy of direction of the velocity of mass m_2 is disrupted in the intervals between collisions. According to this condition, if $\lambda_2 < 0$ then λ_1 must be

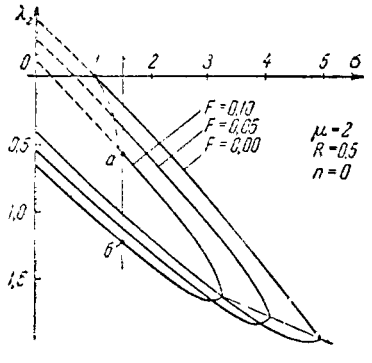

Figure 8.5.

$$\lambda_1 = (2n+1)\pi F\Gamma + \lambda_2 < 0.$$

The boundary line is plotted in accordance with the maximum equality $\lambda_1 = 0$. When R = 1

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inequality (8.9) does not give a direct answer to the question on the maximum value of σ. However, according to (8.8), when R = 1

$$\lambda_2 = -\frac{\mu}{(2n+1)^2\frac{\pi^2}{4}(1+\mu)}\left\{(2n+1)\frac{\pi}{2}\left[\sigma+(2n+1)^2\frac{\pi^2}{4}(1+\mu)\right] \mp \sqrt{(2n+1)^2\frac{\pi^2}{4}-(2n+1)^4\frac{\pi^4}{16}F^2\mu^2}\right\}.$$

From this we can see that in this instance the boundary of the region of actual values is determined by the following inequality: $(2n+1)\frac{\pi}{2}F\mu < 1$.

Finally, let us examine the instance when F = 0. Here, mass m_2 in the intervals between collisions moves with a constant velocity

$$\lambda_2 = \lambda_1 = \lambda = -\frac{(2n+1)\frac{\pi\sigma}{2} \mp \sqrt{(2n+1)^2\frac{\pi^2}{4}+(1-\sigma^2)\left(\frac{1-R}{1+R}\right)^2}}{\left[(2n+1)^2\frac{\pi^2}{4}+\left(\frac{1-R}{1+R}\right)^2\right]\frac{1+\mu}{\mu}}. \qquad (8.10)$$

Here, the region of actual values is limited by the inequality

$$\sigma^2 \leqslant 1+(2n+1)\frac{\pi^2}{4}\left(\frac{1-R}{1+R}\right)^2.$$

Assuming in this inequality that n = 0, 1, 2, 3, ... , we shall find the values of the relative clearance, with which motion conditions can take place, characterized by two collisions in one, three, five, etc., periods of the action of the external force. With clearance values of /265

$$1+\frac{9}{4}\pi^2\left(\frac{1-R}{1+R}\right)^2 \geqslant \sigma^2 \geqslant 1+\frac{\pi^2}{4}\left(\frac{1-R}{1+R}\right)^2$$

motion conditions, characterized by two collisions in three (n = 1), five (n = 2), etc., periods of the action of the external force become possible.

When the size of the clearances is

$$\sigma^2 > 1+\frac{9}{4}\pi^2\left(\frac{1-R}{1+R}\right)^2$$

two collisions will take place in five (n = 2), seven (n = 3) etc., periods of the action of the external force.

Thus, the formal analysis, when F = 0, makes it possible to find the corresponding periodic conditions for a no matter how big clearance value. In reality, with the increase of the clearance value, the forces of friction, present in any real system, have a constantly increasing effect on the motion of a system. The presence of these forces results in the disruption of the

periodic conditions. This is illustrated by the inequality (8.9) which enables us to find the relationships, connecting values n and F.

In this way we have determined the conditions limiting the maximum size of the clearance. We should bear in mind that along with this, conditions exist which limit its maximum size. Indeed, the analysis performed above is based on the assumption that in the interval between collisions, provided for by the periodicity conditions (8.3), no additional collisions occur. No matter how obvious this condition may be, nevertheless it will not reflect in the equations set up. It is precisely this, that explains in particular, that when $\sigma = 0$, value λ, according to formula (8.8), is different from zero, in spite of the fact that the $\sigma = 0$ instance signifies the absence of a clearance ($r = 0$ when $a \neq \infty$) and, it would appear that it should correspond to collision-free motion.

Therefore, establishing the conditions, with which not less than two collisions occur, we should in addition subordinate the system to conditions, during which there were not more than two collisions in the interval examined. /266 These conditions proceed from simple geometric relationships, which are easily established, knowing the laws of motion of both parts of the system in the interval between collisions:

$$x_1 = -\cos(\tau + \varphi) - \left[(2n+1)\frac{\pi}{2} F + \frac{\lambda_2}{\mu}\right]\left[\tau - (2n+1)\frac{\pi}{2}\right],$$

$$x_2 = \frac{F\mu\tau^2}{2} + \left[\tau - (2n+1)\frac{\pi}{2}\right]\lambda_2 - (2n+1)^2\frac{\pi^2}{4} F\mu,$$

$$\tau = 0 \div \pi(2n+1).$$

These laws of motion remain in force also for the second half-period $\tau = \pi(2n + 1)$ to $2\pi(2n + 1)$; it suffices only to reverse the signs.

In Figure 8.6 we are given conventionally the curves of motion of both parts of the system. The shaded area, within the limits of which mass m_2 may move in such a way that in the interval between calculated collisions no additional collisions would occur between it and mass m_1. Evidently, this condition will be fulfilled if

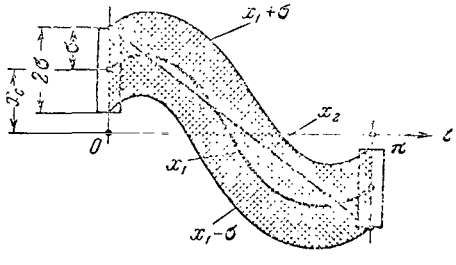

Figure 8.6.

$$|x_2 - x_1| \leq \sigma \qquad (8.11)$$

This inequality should be kept in mind in investigations of every specific motion condition.

In Figure 8.7 we give two different conditions of motion of an element with a clearance, when $n = 0$. To each of these conditions corresponds its own

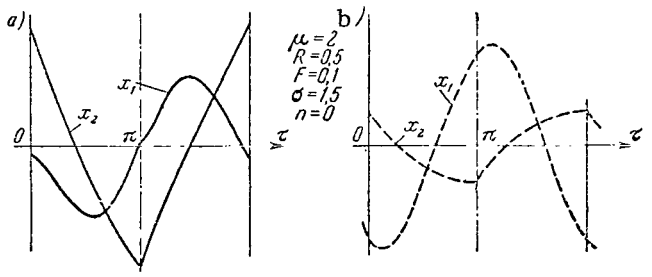

Figure 8.7.

value of λ_2, marked with points a and b respectively on the characteristics of Figure 8.5. /267 Prescribing to ourselves value n = 1, we can determine two other sets of conditions of motions, prescribing n = 2, we determine two more, etc.

Now we shall attempt to answer the question: which of these conditions of motion is actually realized by the system.

§ 8.2. Stability of the Periodic Conditions of Motion.

Here we shall use the same method of finite differences which, in the preceding chapter, gave us the solution of the problem of the stability of motion of the vibrator jumping on the stairs, having generalized this method for the instance of the system with two degrees of freedom.

Let us impart to the dimensionless coordinates and velocities of both parts of the system small perturbations ε_{10}, ε_{20}, ε_{30}, ε_{40}. We write the equations of perturbed motion for a certain ν-th half-period, considering for the sake of definiteness, that it begins after the impact of the right planes. Then, using (8.7), for the perturbed motion, we get:

$$\left.\begin{aligned} x_1^{(\nu)} &= -\cos(\tau + \varphi + \Delta_{(\nu-1)}) + \\ &\quad + (C_1 + \varepsilon_{1(\nu-1)})\tau + (C_2 + \varepsilon_{2(\nu-1)}), \\ x_2^{(\nu)} &= \frac{F\mu\tau^2}{2} + (C_3 + \varepsilon_{3(\nu-1)})\tau + (C_4 + \varepsilon_{4(\nu-1)}). \end{aligned}\right\} \qquad (8.12)$$

Here, as before $\Delta_{(\nu-1)} = \delta_0 + \delta_1 + \ldots + \delta_{(\nu-1)}$ is the accumulated perturbation of the phase of the ν-th impact. Substituting value $\tau = (2n + 1)$ /268 $\pi + \delta_\nu$, in (8.12), we get

$$x_{1e}^{(\nu)} = \cos(\varphi + \Delta_\nu) + (2n+1)\pi C_1 + \delta_\nu C_1 + (2n+1)\pi\varepsilon_{1(\nu-1)} + C_2 + \varepsilon_{2(\nu-1)},$$

$$x_{2e}^{(\nu)} = \frac{F\mu}{2}[(2n+1)^2\pi^2 + 2(2n+1)\pi\delta_\nu] + (2n+1)\pi C_3 + \delta_\nu C_3 + (2n+1)\pi\varepsilon_{3(\nu-1)} + C_4 + \varepsilon_{4(\nu-1)}.$$

Limiting ourselves to the first approximation we find the increments of the coordinates in the perturbed motion by the moment of the $(\nu + 1)$th collision. They will be equal to

$$\Delta x_{1e}^{(\nu)} = -\Delta_\nu \sin\varphi + C_1\delta_\nu + (2n+1)\pi\varepsilon_{1(\nu-1)} + \varepsilon_{2(\nu-1)},$$
$$\Delta x_{2e}^{(\nu)} = F\mu(2n+1)\pi\delta_\nu + (2n+1)\pi\varepsilon_{3(\nu-1)} + \varepsilon_{4(\nu-1)} + C_3\delta_\nu.$$

The speeds of both parts of the system at the ν-th half-period of perturbed motion will be

$$\dot{x}_1^{(\nu)} = \sin(\tau+\varphi+\Delta_{(\nu-1)}) + C_1 + \varepsilon_{1(\nu-1)},$$
$$\dot{x}_2^{(\nu)} = F\mu\tau + C_3 + \varepsilon_{3(\nu-1)}.$$

Substituting value $\tau = (2n+1)\pi + \delta_\nu$ here, we get

$$\dot{x}_{1e}^{(\nu)} = -\sin(\varphi+\Delta_\nu) + C_1 + \varepsilon_{1(\nu-1)},$$
$$\dot{x}_{2e}^{(\nu)} = F\mu[(2n+1)\pi + \delta_\nu] + C_3 + \varepsilon_{3(\nu-1)}.$$

Limiting ourselves to the first approximation, as before, we determine the velocity increments

$$\Delta\dot{x}_{1e}^{(\nu)} = -\Delta_\nu\cos\varphi + \varepsilon_{1(\nu-1)},$$
$$\Delta\dot{x}_{2e}^{(\nu)} = F\mu\delta_\nu + \varepsilon_{3(\nu-1)}.$$

Having determined the increments of dimensionless coordinates and velocities of both masses by the moment of the $(\nu+1)$th collision, we write the laws of their motion at the $(\nu+1)$th half-period. Inasmuch as we have agreed that the ν-th half-period begins after the collision of the right planes, then, consequently, the successive $(\nu+1)$th half-period follows after the collision of the left planes. Having substituted in (8.12) all the signs with the opposite ones, we shall find for this half-period the laws of perturbed motion in the following form: /269

$$\left.\begin{aligned} x_1^{(\nu+1)} &= \cos(\tau+\varphi+\Delta_\nu) - (C_1+\varepsilon_{1\nu})\tau - (C_2+\varepsilon_{2\nu}),\\ x_2^{(\nu+1)} &= -\frac{F\mu\tau^2}{2} - (C_3+\varepsilon_{3\nu})\tau - (C_4+\varepsilon_{4\nu}). \end{aligned}\right\} \qquad (8.13)$$

Differentiating these equations, we find

$$\left.\begin{aligned} \dot{x}_1^{(\nu+1)} &= -\sin(\tau+\varphi+\Delta_\nu) - (C_1+\varepsilon_{1\nu}),\\ \dot{x}_2^{(\nu+1)} &= -F\mu\tau - (C_3+\varepsilon_{3\nu}). \end{aligned}\right\} \qquad (8.14)$$

We substitute value $\tau = 0$ in (8.13) and (8.14) and find, with the

precision down to the small values of the second order, the values of increments of coordinates and velocities of both masses at the beginning of the $(\nu + 1)$th half-period, i.e., at the moment after the $(\nu + 1)$th collision

$$\Delta x_{1\,b}^{(\nu+1)} = -\Delta_\nu \sin\varphi - \varepsilon_{2\nu},\quad \Delta x_{2\,b}^{(\nu+1)} = -\varepsilon_{4\nu},$$
$$\Delta \dot{x}_{1\,b}^{(\nu+1)} = -\Delta_\nu \cos\varphi - \varepsilon_{1\nu},\quad \Delta \dot{x}_{2\,b}^{(\nu+1)} = -\varepsilon_{3\nu}.$$

Now we establish the relationship between the value of perturbations $\varepsilon_{i(\nu-1)}$, $\varepsilon_{i,\nu}$ $(i = 1, 2, 3, 4)$ and Δ_ν in adjacent half-periods of motion, separated by the $(\nu + 1)$th collision, which we shall write in the following manner:

$$\Delta x_{1\,b}^{(\nu+1)} = \Delta x_{1\,e}^{(\nu)};\quad \Delta x_{2\,b}^{(\nu+1)} = \Delta x_{2\,e}^{(\nu)};\quad \Delta x_{1\,b}^{(\nu+1)} = \Delta x_{2\,e}^{(\nu)}. \tag{8.15}$$

In order to fit the velocities of both parts of the system we use the equalities, proceeding from the pulse theorem:

$$\Delta \dot{x}_{1\,b}^{(\nu+1)} = \frac{(1+R)\,\Delta \dot{x}_{2\,e}^{(\nu)} + (\mu - R)\,\Delta \dot{x}_{1\,e}^{(\nu)}}{1+\mu},$$
$$\Delta \dot{x}_{2\,b}^{(\nu+1)} = \frac{(1-\mu R)\,\Delta \dot{x}_{2\,e}^{(\nu)} + \mu\,(1+R)\,\Delta \dot{x}_{1\,e}^{(\nu)}}{1+\mu}.$$

Substituting in (8.15) the previously-found increment values and using the pulse theorem, we obtain a system of five equations in finite differences, by means of which we find the perturbation values sought /270

$$\left.\begin{aligned}
&C_1\Delta_\nu - C_1\Delta_{(\nu-1)} + (2n+1)\,\pi\varepsilon_{1(\nu-1)} + \varepsilon_{2\nu} + \\
&\qquad + \varepsilon_{2(\nu-1)} = 0,\\
&[C_3 + (2n+1)\,\pi F\mu]\Delta_\nu - [C_3 + (2n+1)\,\pi F\mu]\Delta_{(\nu-1)} + \\
&\qquad + (2n+1)\,\pi\varepsilon_{3(\nu-1)} + \varepsilon_{4\nu} + \varepsilon_{4(\nu-1)} = 0,\\
&\qquad -\Delta_\nu \sin\varphi - \varepsilon_{2\nu} + \varepsilon_{4\nu} = 0,\\
&\frac{1+R}{1+\mu}(\cos\varphi + F\mu)\Delta_\nu - \frac{1+R}{1+\mu}F\mu\Delta_{(\nu-1)} + \varepsilon_{1\nu} + \\
&\qquad + \frac{\mu - R}{1+\mu}\varepsilon_{1(\nu-1)} + \frac{1+R}{1+\mu}\varepsilon_{3(\nu-1)} = 0,\\
&\left[-\frac{\mu(1+R)}{1+\mu}\cos\varphi + \frac{1-\mu R}{1+\mu}F\mu\right]\Delta_\nu - \\
&\quad - \frac{1-\mu R}{1+\mu}F\mu\Delta_{(\nu-1)} + \frac{\mu(1+R)}{1+\mu}\varepsilon_{1(\nu-1)} + \varepsilon_{3\nu} + \\
&\qquad + \frac{1-\mu R}{1+\mu}\varepsilon_{3(\nu-1)} = 0.
\end{aligned}\right\} \tag{8.16}$$

As we can see, all the equations are homogeneous and with constant coefficients, dependent on the parameters of the system and perturbation. The characteristic equation, corresponding to this system has the appearance of

$$a_0\beta^4 + a_1\beta^3 + a_2\beta^2 + a_3\beta + a_4 = 0. \tag{8.17}$$

Omitting the intermediate computations, we write the formulas for determining the coefficients

$$\begin{aligned}
a_0 &= \frac{1+\mu}{\mu}\lambda_2 + (2n+1)\pi\frac{2\mu+1}{2}F + \sin\varphi,\\
a_1 &= -\frac{(1-R)(1+\mu)}{\mu}\lambda_2 + (2n+1)\pi\frac{2\mu+1-R}{2}F + \\
&\quad + (3-R)\sin\varphi + (2n+1)\pi(1+R)\cos\varphi,\\
a_2 &= -\frac{(1+R)(1+\mu)}{\mu}\lambda_2 - (2n+1)\pi\frac{2\mu+1+R}{2}F + \\
&\quad + 3(1-R)\sin\varphi + 2(2n+1)\pi(1+R)\cos\varphi,\\
a_3 &= -\frac{(1-R)(1+\mu)}{\mu}\lambda_2 - (2n+1)\pi\frac{2\mu+1-R}{2}F + \\
&\quad + (1-3R)\sin\varphi + (2n+1)\pi(1+R)\cos\varphi,\\
a_4 &= R^2 a_0.
\end{aligned}$$

Now, the problem of the stability of any one periodical set of conditions of a system's motion, as we know, is reduced to the evaluation of the roots of equation (8.17). The motion is stable under the condition that all the roots of β_i (i = 1, 2, 3, 4) with respect to the modulus, are less than one $|\beta_i| < 1$. /271

A detailed analysis of the coefficients of equation (8.17) indicates that they are interconnected with the following relationships

$$\begin{aligned}
a_3 &= 4a_0 + 2a_2 - 3a_1,\\
a_4 &= 3a_0 + a_2 - 2a_1.
\end{aligned}$$

After substituting these relationships, our characteristic equation assumes the following appearance:

$$(\beta+1)^2[a_0\beta^2 + (a_1 - 2a_0)\beta + a_4] = 0.$$

In this way, two roots prove equal $\beta_{1,2} = -1$, in other words, the analysis of stability resulted in a doubtful case when the roots of the characteristic equation were found equal to extreme values corresponding to the limit of stability of the motion conditions examined. It would appear that this result exhausts the problem of the stability analysis by the method used by us, inasmuch as for solving such a doubtful instance it is required to perform an

analysis of stability in the second approximation.

In reality, however, the matter is quite different. We should remember that the two-mass system examined by us is free of any relationships whatever, which would restrain it from any movement, no matter how big, along x axis (see Figure 8.4). Having this in view we can make the assumption that as the result of the action of random perturbations, the system as a whole may prove unstable and may go as far away as needed from the position selected as the initial one, and that precisely this circumstance was reflected in the fact that the two roots of the characteristic equation were found equal to $\beta_{1,2} = -1$. But then it should not have an effect on the stability of the relative motion of both parts of the system, and it is precisely here, that the relative motion interests us first of all. In order to verify the correctness of our assumption, we shall first determine the remaining roots of equation (8.17). /272 Then, we shall solve directly the problem of the stability of the relative motion of both parts of the system. A comparison of the results obtained by the two different methods will enable us to answer the question of the stability of the periodic conditions of motion.

For determining the other two roots of equation (8.17) we use equation

$$a_0\beta^2 + (a_1 - 2a_0)\beta + a_4 = 0. \tag{8.18}$$

Using Schur's criteria, we shall establish the regions of the parameter values, for which the roots of this equation are less than one with respect to the modulus. For the time being we shall conventially call these regions the "stability regions" of the corresponding conditions of motion. We know that beyond their limits no stable conditions of the type examined exist.

Substituting the values of coefficients a_0, a_1 and a_4 in (8.18) we get

$$\begin{aligned}\left[\lambda_2 + (2n+1)\frac{\pi}{2}F\mu\right]\beta^2 + \left[-2R\lambda_2 + (2n+1)\frac{\pi}{2}\frac{\mu(1+R)^2}{1+\mu}\times\right.\\ \times\cos\varphi + (2n+1)\frac{\pi}{2}F\mu\left.\frac{\mu(1-R)^2 - 2(R+\mu)}{1+\mu}\right]\beta +\\ + R^2\left[\lambda_2 + (2n+1)\frac{\pi}{2}F\mu\right] = 0.\end{aligned} \tag{8.19}$$

The following inequalities serve as the necessary and sufficient conditions for making the roots of this equation less than one with respect to the modulus:

$$\left|\frac{a_4}{a_0}\right| < 1, \qquad \left|\frac{a_1 - 2a_0}{a_0 + a_4}\right| < 1. \tag{8.20}$$

Inasmuch as $a_4/a_0 = R^2$, the first of these inequalities is always

fulfilled, with the exception of the instance when R = 1. For any real system R is always less than 1, and therefore, this limitation is immaterial.

In order to determine the parameters of the system, satisfying the second inequality, we shall first examine the exact equalities

$$\frac{a_1-2a_0}{a_0+a_4}=1, \tag{8.21}$$

$$\frac{a_1-2a_0}{a_0+a_4}=-1. \tag{8.22}$$

From equation (8.21) it follows that /273

$$-a_0+a_1+a_4=0.$$

Substituting the values of a_0, a_1 and a_4 here we get the following equality:

$$2(1-R)\sin\varphi+(2n+1)\pi(1+R)\cos\varphi=0.$$

The values of sin ϕ and cos ϕ as functions of the parameters of the system and actuation were determined in the preceding paragraph. Substituting them in the last equation, we find that

$$(2n+1)^2\frac{\pi^2}{4}+\left(\frac{1-R}{1+R}\right)^2=\left[-(2n+1)^2\frac{\pi^2}{4}F\mu-\frac{1-R}{1+R}\sigma\right]^2,$$

from which, taking into account that σ is a substantially positive value, we find

$$\sigma=\frac{1+R}{1-R}\left[\sqrt{(2n+1)^2\frac{\pi^2}{4}+\left(\frac{1-R}{1+R}\right)^2}-(2n+1)^2\frac{\pi^2}{4}F\mu\right]. \tag{8.23}$$

We select certain fixed values of μ and F. Then curves $\sigma=\sigma(R)$, calculated with this formula for various values of n, represent one of the boundaries of the stable conditions of the system's motion. On this boundary, at least one root of equation (8.18) is equal to $|\beta|=1$. The problem of on which side of the boundary are the stability regions located, is solved by the direct verification, through substitution in inequality (8.20) of values σ, which differ little from values calculated according to (8.23).

We find that to condition $|\beta|<1$ corresponds the values

$$\sigma<\frac{1+R}{1-R}\left[\sqrt{(2n+1)^2\frac{\pi^2}{4}+\left(\frac{1-R}{1+R}\right)^2}-(2n+1)^2\frac{\pi^2}{4}F\mu\right]. \tag{8.24}$$

Comparing (8.24) with (8.9) we find that the boundary of the regions of periodic conditions stability, determined in this way, coincides with the boundary of the regions of their existence. We shall find the second boundary of the regions of stability by using equation (8.22). Substituting the values of a_0, a_1 and a_4 in it we get the following equation: /274

$$\lambda_2=(2n+1)\pi\frac{\mu}{(1+\mu)\left[2-(2n+1)^2\frac{\pi^2}{2}\right]}\times\left\{\sigma+\left[(2n+1)\frac{\pi^2}{4}(1+\mu)-\frac{1+R+2\mu}{1+R}\right]F\right\}. \quad (8.25)$$

Using (8.8), we get the quadratic equation

$$\left[A_3^2+\left(\frac{1-R}{1+R}\right)^2\right]\sigma^2-\left[2A_3A_4-(2n+1)^2\frac{\pi^2}{2}\mu\frac{1-R}{1+R}F\right]\sigma+(2n+1)^4\frac{\pi^4}{16}F^2\mu^2-A_1+A_4^2=0. \quad (8.26)$$

Here

$$A_1=(2n+1)^2\frac{\pi^2}{4}+\left(\frac{1-R}{1+R}\right)^2,$$

$$A_2=\frac{(1-R)(2\mu R+R-1)}{(1+R)^2}-(2n+1)^2\frac{\pi^2}{4}(1+\mu),$$

$$A_3=(2n+1)\frac{\pi}{2}\left[1+\frac{A_1}{1-(2n+1)^2\frac{\pi^2}{4}}\right],$$

$$A_4=(2n+1)\frac{\pi}{2}FA_2+\frac{(2n+1)\pi}{2-(2n+1)^2\frac{\pi^2}{4}}\times$$

$$\times\left[\frac{1+R+2\mu}{1+R}-(2n+1)\frac{\pi^2}{4}(1+\mu)\right]FA_1.$$

Having prescribed certain values of μ and F, we get from equation (8.26) the boundary values of $\bar{\sigma}$ as a function of R for various values n = 0, 1, 2, ... To the boundary values of $\bar{\sigma}$ corresponds the root of the characteristic equation β = -1. Direct verification indicates that $\bar{\sigma}=\bar{\sigma}(R)$ curves from the lower boundaries of the regions of stability of the periodic conditions. In this way, the regions of the values of the system's parameters, corresponding to the stable conditions of the system's motion, are found to be limited by the following inequalities: /275

$$\bar{\sigma}<\sigma<\frac{1+R}{1-R}\left[\sqrt{(2n+1)^2\frac{\pi^2}{4}+\left(\frac{1-R}{1+R}\right)^2}-(2n+1)^2\frac{\pi^2}{4}F\mu\right]. \quad (8.27)$$

In accordance with these inequalities, a stability chart for n = 0 and several values of the value of force F is plotted in Figure 8.8. As we can see, the presence of the force of friction results, in this case, to a certain expansion of the region of stability, but does not eliminate the possibility of formation of unstable conditions. Point A in Figure 8.8 corresponds to the

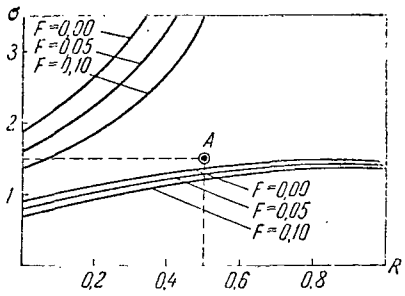

Figure 8.8.

values of the parameters for which the laws of motion are plotted in Figure 8.7. (We remind the reader that the solution of the problem of the stability of a certain set of conditions of motion should be preceded by its verification with inequalities (8.11)). The analysis of stability fulfilled by us enables us now to answer the question about which of the two possible conditions will be realized by the system. To each of them corresponds the specific value of λ_2, calculated in accordance with formula (8.8). On the other side, these λ_2 values are used directly in the determination of the lower boundaries of the regions of stability according to equation (8.25). Substituting successively here the values λ_{21} and λ_{22}, corresponding to signs $\mp$ in formula (8.8), we may convince ourselves that Schur's criteria are satisfied with the value of λ_2, corresponding to the minus sign before the radical. In other words, those motion conditions of a system are found to be stable, which are accompanied by the more "active" impact interaction of its parts. In Figure 8.7 these motion conditions are plotted with solid lines.

We have made the same deduction in the preceding chapter, examining the motion of the vibrator jumping on the stairs. There too, of the two possible conditions, corresponding to each of the values n = 0, n = 1, n = 2, ..., the one that proves stable is the one under which the velocity of the vibrator's impact against the step proves greater. /276

We may express the assumption that in the case where the parameters of a system are such that several conditions of motion of different multiplicity (n = 0, 1, 2, ... for a vibrator jumping on steps, or n = 1, 2, 3, ... for a nonlinear element with a clearance) occur, the system realizes that one of the possible conditions, to which corresponds the maximum velocity of the impact or collision. However, the question of the correctness of the assumption expressed remains open.

§ 8.3. Stability in a Relative Motion.

Now we are to resolve the doubts which had arisen in connection with the fact that two roots of a characteristic equation (8.17) proved equal to $\beta_{1,2} = -1$. In the preceding paragraph the assumption was expressed that this may be done, excluding the transfer motion of the system as a whole, and examining the stability of only the relative motion of its parts, coupled with a clearance (Figure 8.9). Using (8.7), we shall write the law of relative motion in the following way:

$$x = x_1 - x_2 = -\cos(\tau + \varphi) - \frac{F\mu\tau^2}{2} + \bar{C}_1\tau + \bar{C}_2. \qquad (8.28)$$

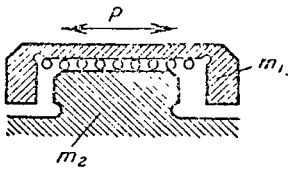

Figure 8.9.

The integration constants of $\bar{C}_1$ and $\bar{C}_2$ are expressed by values C_1, C_2, C_3 and C_4, determined in §8.1:

$$\bar{C}_1 = C_1 - C_3, \quad \bar{C}_2 = C_2 - C_4.$$

Thus, the law of relative motion is fully determined, as before in the time interval $\tau = 0$ to $(2n + 1)\pi$. Because of the symmetry, the same equation (8.28) works also in the second half of the relative motion, under the condition of inversion of all the signs in it. /277

We introduce a perturbation in the relative motion of both parts of the system and write the equation of the perturbed motion for the ν-th half-period. Just as in the previous paragraph, we shall consider that it begins after the impact of the right surfaces

$$x^{(\nu)} = -\cos(\tau + \varphi + \Delta_{(\nu-1)}) - \frac{F\mu\tau^2}{2} + (\bar{C}_1 + \varepsilon_{1(\nu-1)})\tau + (\bar{C}_2 + \varepsilon_{2(\nu-1)}).$$

Substituting value

$$\tau = (2n + 1)\pi + \delta_\nu,$$

here, we shall get the value of the relative coordinate in the perturbed motion by the moment of the $(\nu + 1)$th collision $-x_e^{(\nu)}$. Then substituting value $\tau = (2n + 1)\pi$, in (8.28) we shall get value x_e in the unperturbed state. The difference $\Delta x_e^{(\nu)} = x_e^{(\nu)} - x_e$ will represent the sought increment of the coordinate at the moment of the $(\nu + 1)$th collision. It is equal to

$$\Delta x_e^{(\nu)} = [-(2n + 1)\pi F\mu - \sin\varphi + \bar{C}_1]\Delta_\nu + [(2n + 1)\pi F\mu - C_1]\Delta_{(\nu-1)} + \varepsilon_{2(\nu-1)} + (2n + 1)\pi\varepsilon_{1(\nu-1)}.$$

The relative velocity in the perturbed motion is expressed as follows:

$$\dot{x}^{(\nu)} = \sin(\tau + \varphi + \Delta_{(\nu-1)}) - F\mu\tau + \bar{C}_1 + \varepsilon_{1(\nu-1)}.$$

Accordingly, the velocity increment in the relative motion will be

$$\Delta\dot{x}_e^{(\nu)} = -\Delta_\nu\cos\varphi - F\mu\Delta_\nu + F\mu\Delta_{(\nu-1)} + \varepsilon_{1(\nu-1)}.$$

For the perturbed motion in the next $(\nu + 1)$th interval, beginning after the collision of the left planes, we shall have

$$x^{(\nu+1)} = \cos(\tau + \varphi + \Delta_\nu) + \frac{F\mu\tau^2}{2} - (\bar{C}_1 + \varepsilon_{1\nu})\tau - (\bar{C}_2 + \varepsilon_{2\nu}),$$

$$\dot{x}^{(\nu+1)} = -\sin(\tau + \varphi + \Delta_\nu) + F\mu\tau - \bar{C}_1 - \varepsilon_{1\nu}.$$

Substituting value $\tau = 0$ here, we shall get the coordinate and velocity in the perturbed motion at the moment after the $(\nu + 1)$th collision -- $x_b^{(\nu+1)}$ and $\dot{x}_b^{(\nu+1)}$. Substracting their value in the unperturbed motion we get the increments sought /278

$$\Delta x_b^{(\nu+1)} = -\Delta_\nu \sin\varphi - \varepsilon_{2\nu},$$
$$\Delta \dot{x}_b^{(\nu+1)} = -\Delta_\nu \cos\varphi - \varepsilon_{1\nu}.$$

We shall now form the conditions of fitting two adjacent perturbed motion intervals in such a way, as if mass m_2 of the system examined were immobile (see Figure 8.9). Here

$$\Delta x_e^{(\nu)} = \Delta x_b^{(\nu+1)},$$
$$\Delta x_b^{(\nu+1)} = 0,$$
$$\Delta \dot{x}_b^{(\nu+1)} = -R\Delta \dot{x}_e^{(\nu)}.$$

Taking into account the values found above, we get the following homogeneous system of equations in finite differences:

$$\begin{aligned}
&[-(2n+1)\pi F\mu - \sin\varphi + \overline{C_1}]\,\Delta_\nu + [(2n+1)\pi F\mu - \overline{C_1}]\,\Delta_{(\nu-1)} + \\
&\quad + (2n+1)\pi\varepsilon_{1\,(\nu-1)} + \varepsilon_{2\,(\nu-1)} = 0, \\
&\quad \Delta_\nu \sin\varphi + \varepsilon_{2\nu} = 0, \\
&[(R+1)\cos\varphi + F\mu R]\,\Delta_\nu - F\mu R\Delta_{(\nu-1)} + \varepsilon_{1\nu} - R\varepsilon_{1\,(\nu-1)} = 0.
\end{aligned}$$

Substituting $\overline{C}_1 = C_1 - C_3$ here, and applying the method of solution of such systems already known to us, we find the corresponding characteristic equation. It will represent a quadratic equation of the following appearance:

$$\begin{aligned}
&\left[\lambda_2 + (2n+1)\frac{\pi}{2}F\mu\right]\beta^2 + \left[-2R\lambda_2 + (2n+1)\frac{\pi}{2}\times\right. \\
&\left.\times\frac{\mu(1+R)^2}{1+\mu}\cos\varphi + (2n+1)\frac{\pi}{2}F\mu\frac{\mu(1-R)^2 - 2(\mu+R)}{2(1+\mu)}\right]\beta + \\
&\quad + R^2\left[\lambda_2 + (2n+1)\frac{\pi}{2}F\mu\right] = 0. \qquad (8.29)
\end{aligned}$$

Comparing this equation with equation (8.19) we discover their complete identity. In this way we find the verification of the assumption that the presence of roots, identically equal to minus 1 ($\beta_{1,2} + -1$), is connected with the unlimited nature of the transfer motion of a system. In examining the relative motion, we find that the order of the characteristic equation decreases, /279

and precisely those roots disappear, which made us doubt the answer to the question about the stability of periodic conditions.

In concluding the analysis of the stability of motion of a nonlinear element with a clearance, let us examine an instance, when friction is absent (F = 0). Here, the characteristic equation assumes the following appearance:

$$\lambda_2\beta^2+\left[-2R\lambda_2+(2n+1)\frac{\pi}{2}\frac{\mu(1+R)^2}{1+\mu}\cos\varphi\right]\beta+R^2\lambda_2=0.$$

Dividing by λ_2 we get

$$\beta^2-\left[2R+(2n+1)\frac{\pi}{2}(1-R^2)\operatorname{ctg}\varphi\right]\beta+R^2=0.$$

Inequalities (8.27), limiting the regions of stable solutions here are:

$$\sqrt{\frac{A_1}{A_3^2+\left(\frac{1-R}{1+R}\right)^2}}<\sigma<\frac{1+R}{1-R}\sqrt{(2n+1)^2\frac{\pi^2}{4}+\left(\frac{1-R}{1+R}\right)^2}. \qquad (8.30)$$

As we can see, when F = 0, among the three constant parameters of the systems, R, σ, and μ the stability of the motion conditions examined is affected only by R and σ, and the stability is independent of the mass ratio of the parts. In accordance with inequalities (8.30), in Figure 8.10, for values n = o, n = 1, n = 2 we have plotted a stability chart, which divides the plane of parameters (R, σ) into a series of regions. We have shaded the areas within the bounds of which the motion conditions are stable with n = 0 or n = 1, or n = 2. In the right-hand part of the chart we designate regions within the bounds of which two (n = 0 and n = 1, n = 1 and n = 2) and even three (n = 0, n = 1, n = 2) sets of conditions are stable simultaneously. However, there are cases, with considerably large values of the relative clearance, when even a small friction has a substantial effect on the picture of motion of a system and results in the disruption of its periodicity.

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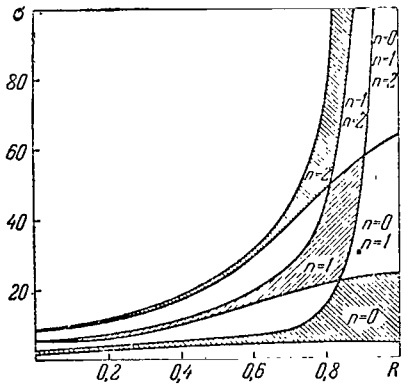

Figure 8.10.

The results of the analysis of the stability of periodic conditions carried out were supported by experiments performed according to the following system (Figure 8.11). On the table 1 of the vibrating testing device, which performs a harmonic motion, two steel stops 2 are installed, the distance between which may be regulated: the table of the vibrating test device with its stops, simulated mass m_1 of the system

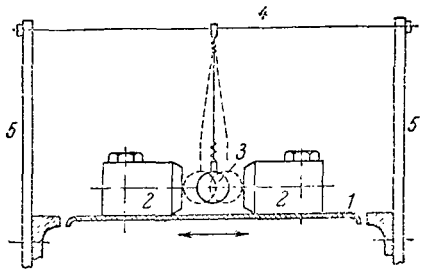

Figure 8.11.

examined. The role of the second part of the system was played by a steel ball 3, suspended on a nylon thread, the upper end of the thread is equipped with a ring, set freely on a thin steel rod 4. Rod 4 is fixed on stationary supports 5. This design has been selected in order to reduce to zero the forces of resistance acting on the ball. Earlier, it was shown that the ratio between the masses of a system has no effect on the distribution of stability regions.

In the process of carrying out the experiment, the movement of the table carrying the stops, and the moments of collisions between the ball and the supports were recorded. In Figure 8.12 /281 we give several oscillograms taken with various values of the relative clearance. The curve in the central part of every picture represents the movement of the table. The current splashes on the upper and lower straight lines correspond to the moments of collision of the ball with one or the other support.

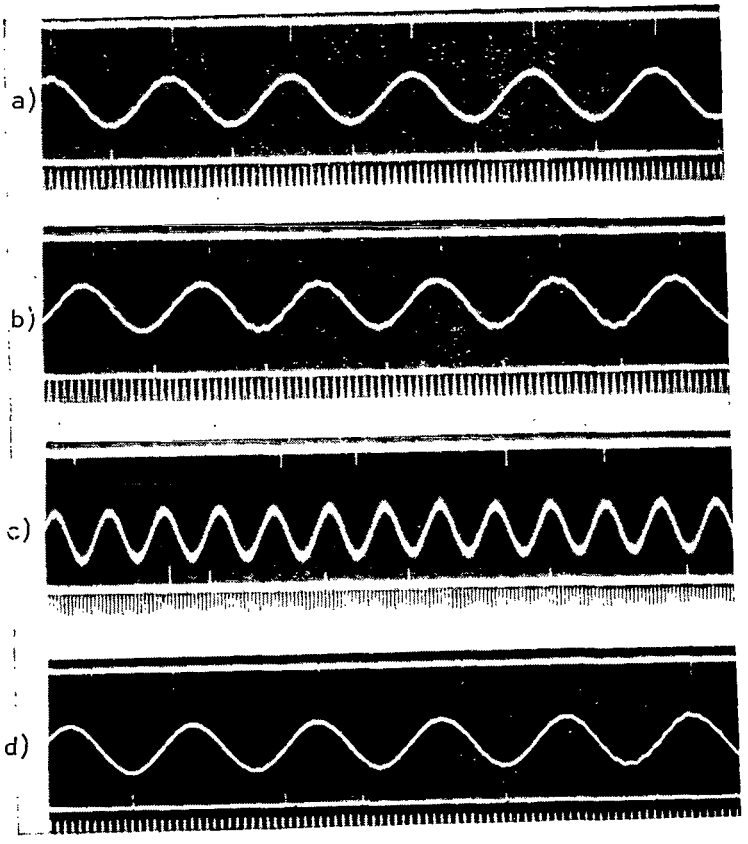

Figure 8.12.

Values $\sigma = 4$ and $\sigma = 5$ /282 correspond to pictures a) and b). As we can see, with such clearances there are stable periodic motions, in the process of which two collisions take place in one period of the action of the external force (condition $n = 0$). Having evaluated the coefficient of restitution after an impact measuring $R = 0.55$, corresponding to the impact of the steel balls, we shall turn to the stability chart, the corresponding sector of which on an increased scale is given in Figure 8.13. As should have been expected, both points a and b are located in the $n = 0$ stability region. Then, the size of the clearance was increased to $\sigma = 6$. Here, the ball's motion became aperiodic, its collisions with the supports acquired a

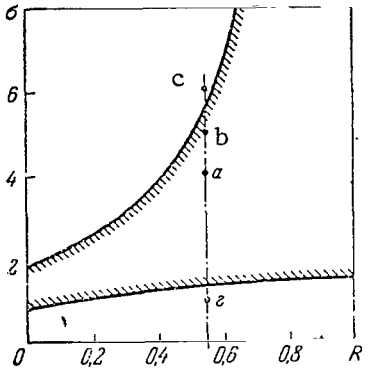

Figure 8.13.

chaotic character, as shown by oscillogram 8.12, c. The cause of this becomes understandable if we take into account that in chart 8.13 point c is found outside the stability region. Finally, in Figure 8.12,d, one more oscillogram is given, taken when $\sigma = 1$. The result of this experiment was unexpected. In spite of the fact that point d characterizing these conditions in chart 8.13, is located outside the stability region, the movement of the ball proved periodical. However, in this case it moved not as it did in the preceding cases when its movement was also periodical. Having examined the picture attentively, we may find that to three periods of movement of the vibration table correspond six impacts of the ball against the supports, i.e., two impacts per period, as before; the impacts of the ball against the left and right supports alternate as before, however, in this case the time intervals between the impacts change periodically. Thus, the fact that the characteristic point on the stability chart is located outside the bounds of the stability regions, calculated for conditions of the same type, /283 does not yet prove that a system with such parameter values will not make periodic motions at all.

We should state that the existing methods do not enable us to detect all the possible periodic conditions, and in this way solve to the end the problem of the dynamics of the settled motion of the vibro-impact system. Nevertheless, the results obtained in this chapter, enabled us to discover a number of new properties and features, inherent in systems with a clearance. Thus, in particular, using the laws of motion (8.11) and knowing the regions of the values of parameters, within the bounds of which these laws apply, we can accurately or approximately determine the transmission functions for the nonlinear element with a clearance, account for the effect of the clearance on the dynamics of the system as a whole, evaluate its dissipative properties, etc.

The investigations performed will be incomplete if we do not examine the practically important instance when the collisions of a system's part are inelastic, so that the restitution coefficient $R = 0$. This instance was already mentioned in the preceding chapter, and we shall now examine it in detail, as applied to a two-mass system with a clearance.

§ 8.4. Absolutely Inelastic Collisions ($R = 0$).

Newton's hypothesis, according to which the coefficient of restitution upon an impact, depends only on the properties of the materials of the colliding bodies, and does not depend on their shape and collision velocity, was revised substantially during the last decades (see, for example [16] and the literature quoted there). Experiments indicate, that even in such a simple case as the impact of a ball against a plane, the value of the coefficient of restitution varies within a wide range. The problems of collision of bodies

possessing flat or cylindrical surfaces, have been investigated very little up to the present time, and we have not been able to find any data on the determination of the corresponding coefficients of restitution in the literature. However, on the basis of the works already performed, we can claim, that for real kinematic pairs, the coefficient of restitution depends to a great extent both on the velocity of the collision and the shape of the elements of the kinematic pair, and on the presence and thickness of the layer of lubricant, the state of the colliding surfaces, and technological errors in their manufacture. /284

In the present book we do not assign to ourselves the task of the theoretical or experimental substantiation of the above-mentioned claim. The experiments described below were performed only with the purpose of demonstrating the fact that in the evaluation of the value of the coefficient of restitution upon an impact, we cannot always use the tabular values, that often the collisions of the elements of a kinematic pair may be considered inelastic, and that because of this the R = 0 instance possesses not only a theoretical but also a practical interest.

In Figure 8.14 we give a diagram of an experimental model, installed on a vibration table. It represents a body 1, carrying two flat stops 2, made of bronze, and a spindle 3. Striker 4 with flat end surfaces is set loosely on the spindle. One of the stops 2 is made immobile, as the result of which the value of the per unit clearance between the striker and the stops may be regulated within sufficiently-wide limits. Spindle 3 and the immobile stop are insulated from the body and from one another and are included in a measuring system, making it possible to record the moments of the striker's contact with one or the other stop. The end surfaces of the striker and stops were covered with vaseline oil, and then wiped until the disappearance of visible traces of lubricant. The model was installed on a vibrator table and subjected to the action of harmonic vibration with the amplitude ρ_0 = 1.0mm and frequency ω = 220 1/sec, which corresponds to the maximum vibration velocity v = 0.22 meter/sec.

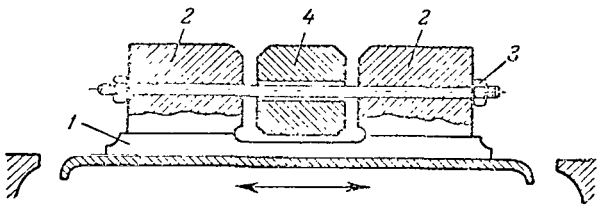

Figure 8.14.

In Figure 8.15 we present the typical oscillograms recorded for a steel (8.15,a) and bronze (8.15,b) strikers with different clearance values. The speed of the vibrator table is recorded on the central part of each oscillogram. The broken lines in the upper and lower parts of the pictures correspond to the absence of the contact between the striker and one of the stops; the impact moments are indicated by the splashes of current. /285

As indicated by the oscillograms, the movement of the striker has a periodic character. Coming into contact with one of the stops the striker appears to "stick" to it; the subsequent "breaking away" of the striker occurs only after

the speed of the vibration table reaches its maximum value, and then follows the period of free motion of the striker in the gap, after which it comes into contact with the other stop, and breaks away from it, again with the maximum velocity of the vibrator. The effect of connection of the striker to the supports may take place only if the coefficient of restitution upon impact differs little from zero. Such a periodic motion of the striker was maintained in the experiments described for the relative clearance value range of $\sigma = 0.0$--1.2. Beyond the limits of this range, splashes appear on the oscillograms, indicating the disruption of the periodicity of the striker's motion, which points to the fact, that for real structures, the coefficient of restitution upon impact, depends on the collision velocity.

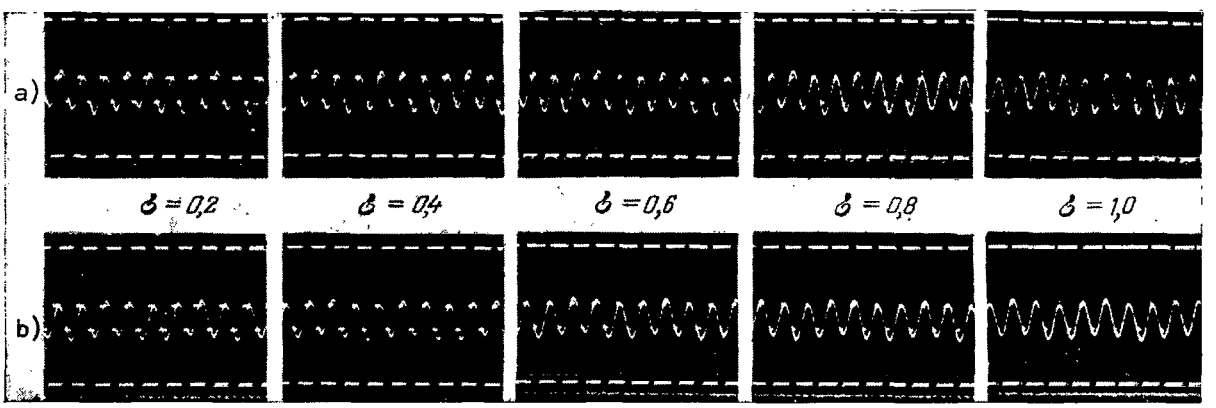

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Figure 8.15.

On the basis of the experiments performed we shall not make any generalizations pertaining to the values of coefficients of restitution upon the impact of flat surfaces. We shall note only that, as indicated by experiments, in the collisions of elements of the real kinematic pairs, the coefficient of restitution is much smaller than that in a direct central impact of balls, made of the same material; with relatively small oscillation velocities its value may be even close to zero.

We shall remind the reader, that in the preceding paragraph the value of the coefficient of restitution was assumed to be $0 < R \leq 1$. The $R = 0$ instance was excluded from the examination because in an absolutely inelastic impact, such motion conditions are possible, with which the relationships determined there, become invalid.

As we know, in the case of inelastic collision, the velocities of both parts of a system, at the moment after the collision, prove equal. In this case, depending on the parameters of the systems, two entirely different conditions of its periodic motions may take place. If, after collision, the velocity of mass m_1, which is under the effect of an external force, decreases, both parts of the system move apart; here all the previously-performed analysis remains in force. In the opposite case, i.e., if after the collisions the velocity of mass m_1 increases, both masses move jointly for a certain period of time; the external force in this interval of time acts on both masses (mass m_2

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"joins" mass m_1) and equations (8.2) as well as conditions (8.3), prove insufficient for describing the full period of the system's motion.

Turning to the examination of the R = 0 instance, we shall first determine under which conditions the systems of motion exist which are accompanied by "connection" of mass m_2 to mass m_1. In order to simplify the analysis we assume that F = 0. Differentiating the first one of equations (8.7) we find the velocity of mass m_1 equal to

$$\dot{x}_1 = \sin(\tau + \varphi) - \frac{\lambda}{\mu}.$$

We remind the reader that according to the adopted system of count of dimensionless time, at the moment of impact $\tau = 0$. We determine the velocity of mass m_1 for the time moment differing from the impact moment by a small value-$\Delta\tau$. Assuming that $\sin \Delta\tau = \Delta\tau \cos \Delta\tau = 1$, we find the velocity of mass m_1 equal to

$$\dot{x}_1 = \Delta\tau \cos\varphi + \sin\varphi - \frac{\lambda}{\mu}.$$

Substituting here the previously-found values of $\sin\phi$ and $\cos\phi$ when F = 0 and R = 0, we get

$$\dot{x}_1 = \Delta\tau \left[\sigma + (2n+1)\frac{\pi}{2}\frac{1+\mu}{\mu}\lambda\right] + \lambda.$$

According to (8.10) the dimensionless velocity λ after the collision of the right-hand planes, is always $\lambda < 0$. Consequently, if

$$\sigma \geqslant (2n+1)\frac{\pi}{2}\frac{1+\mu}{\mu}\lambda,$$

the velocity of mass m_1 after the collision, begins to decrease in its absolute value, and both parts of the system move apart. Here, the maximum magnitude of value σ is still limited by inequality (8.9) when R = 0 and F = 0 are substituted in it. /288

In the case when

$$\sigma < (2n+1)\frac{\pi}{2}\frac{1+\mu}{\mu}\lambda,$$

there are conditions accompanied by periodic connection of mass m_2 to m_1. It is not difficult to show that with the limit value of the size of the relative clearance equal to

$$\sigma = (2n+1)\frac{\pi}{2}\frac{1+\mu}{\mu}\lambda,$$

both parts of the systems move apart, inasmuch as the sign of the acceleration of mass m_1 proves opposite to the sign of its velocity. By substituting the value of λ from (8.10) into the last equation we find that

$$\sigma = (2n+1)\frac{\pi}{2}.$$

We shall note, incidentally, that with the other value of σ the other values, characterizing the periodic motion conditions described, will be found equal to

$$\lambda = \frac{\mu}{1+\mu}, \quad \varphi = \frac{3}{2}\pi, \quad x_c = -(2n+1)\frac{\pi}{2}\frac{1}{1+\mu}.$$

Thus, when R = 0 and when $\sigma \geq (2n+1)\frac{\pi}{2}$ the entire preceding analysis, including the analysis of stability of all manner of periodic motion conditions, remains in force.

Now, we shall set up the conditions of periodicity, encompassing the conditions of motion with connection of masses. For this, we shall first of all introduce value τ_0, characterizing the length of the period of joint motion of both masses after their collision. To determine this value in the future we shall use the concept, which is evident from the preceding discussions, that the limit of the time interval of the joint motion of both masses coincides with the moment when the velocity of their motion will reach a maximum. Following this, the velocity of mass m_2 remains constant, while /289 the velocity of mass m_1 begins to decrease.

We shall in the future investigate only the periodic conditions of motion, and only those, in the process of which the collisions of the right and left planes of both elements of the kinematic pair (m_1 - m_2) alternate.

Now, each of the two identical half-periods forming a full period of a system's motion, is broken down into two intervals. We write out the boundary conditions on these intervals, adopting, as before, for the beginning of the count, the collisions of the right-hand elements, and using the dimensionless form of recording

$$\left.\begin{aligned} &x_1 = x_c, \quad x_2 = x_c + \sigma, \quad \dot{x}_1 = \dot{x}_2 = v_2 \text{ when } \tau = 0, \\ &x_1 = x_0, \quad x_2 = x_0 + \sigma, \quad \dot{x}_1 = \dot{x}_2 = \lambda \text{ when } \tau = \tau_0, \\ &x_1 = x_c, \quad x_2 = -(x_c + \sigma), \quad \dot{x}_1 = v_1, \quad \dot{x}_2 = \lambda \text{ when } \tau = (2n+1)\pi. \end{aligned}\right\} \qquad (8.31)$$

In the τ = 0 to τ_0 time interval both parts of the system move jointly under the effect of an external force. The laws of their motion, by analogy with (8.7) will be

$$\left.\begin{aligned} x_{1,2} &= -\frac{\mu}{1+\mu}\cos(\tau+\varphi)+C_1\tau+C_2, \\ \dot{x}_{1,2} &= \frac{\mu}{1+\mu}\sin(\tau+\varphi)+C_1. \end{aligned}\right\} \tag{8.32}$$

In the second time interval $\tau = \tau_0$ to $(2n + 1)\pi$ the laws of motion of both parts of the system have the following appearance:

$$\left.\begin{aligned} x_1 &= -\cos(\tau+\varphi)+C_3\tau+C_4, \\ x_2 &= (x_0+\sigma)-\lambda(\tau-\tau_0), \\ \dot{x}_1 &= \sin(\tau+\varphi)+C_3; \quad \dot{x}_2 = -\lambda. \end{aligned}\right\} \tag{8.33}$$

Substituting the periodicity conditions (8.31) in (8.32) and (8.33), we can find the values C_1, C_2, C_3, C_4, determining the periodic conditions of the type examined

$$C_1 = 0, \quad C_2 = -\frac{\sigma}{1+\mu}, \quad C_3 = \frac{1}{1+\mu}, \quad C_4 = -\frac{\sigma\mu}{1+\mu} - \frac{3\pi-2\varphi}{2(1+\mu)}.$$

Conditions (8.31) and laws of motion (8.32) and (8.33) remain in force also for the second half-period of motion under the condition of reversal of all their signs. /290

It is not hard to prove that the velocity of the combined motion of both masses attains a maximum when

$$\tau_0 = \frac{3\pi-2\varphi}{2}.$$

The value of ϕ is determined from the following transcendent equation:

$$\cos\varphi+\varphi = 2\sigma-\left(2n-\frac{1}{2}\right)\pi. \tag{8.34}$$

In Figure 8.16 we give the curves of $\phi = \phi(\sigma, n)$ for values $n = 0$ and $n = 1$. In the

$$\frac{\pi}{2}\leqslant\sigma\leqslant\pi, \quad \frac{3}{2}\pi\leqslant\sigma\leqslant 2\pi, \quad (2n+1)\frac{\pi}{2}\leqslant\sigma\leqslant(n+1)\pi$$

value ranges, there is no connection of masses, and therefore all the relationships determined in the preceding paragraph remain in force.

Now we shall examine briefly the problem of the dynamic stability of the

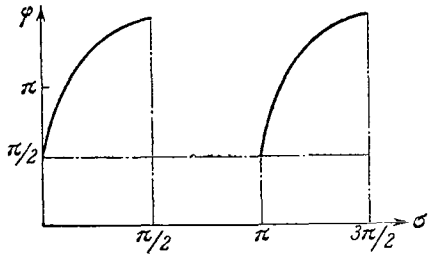

Figure 8.16.

periodic conditions of motion of a system, established with the R = 0 values.

For the

$$(2n+1)\frac{\pi}{2} \leqslant \sigma \leqslant \sqrt{1+(2n+1)\frac{\pi^2}{4}}$$

value ranges this problem has been already solved in the preceding paragraph (see Figures 8.8 and 8.10). We remark only, that/291 there it was solved without taking into consideration the circumstance that with values $\sigma < (2n + 1)\frac{\pi}{2}$ special conditions are established, accompanied by connection of masses (R = 0), in the same way as it had been solved without taking into account the possibility of additional collisions (inequalities (8.11)).

As for the relative clearance regions

$$n\pi < \sigma < (2n+1)\frac{\pi}{2},$$

corresponding to the conditions of motions with connected masses, the results obtained earlier are already inapplicable to the analysis of the stability of these conditions. However, this problem is solved with relative simplicity, on the basis of the examination of the kinematic picture of perturbed motion. In the perturbed state the phase of the successive collision may differ from the calculated one. However, independently of the variation of this phase, both masses after collision will always move jointly until the moment of achievement of the maximum velocity. Evidently, with all these conditions the perturbed motion will be performed in the vicinity of the unperturbed, and consequently the corresponding conditions of motion will be stable. This deduction is also supported by the experiments the results of which are given in Figure 8.15.

In conclusion of this paragraph we note that in investigating mechanisms with elastic couplings and control systems we encounter instances when along with the clearance we have to take into account also the elasticity of one or both parts of the system. Thus, in particular, in work [27] along with the nonlinear elements of the "clearance" type a nonlinear element for the "fork" type is examined, the dynamic model of which is given in Figure 8.17,a. As another example we may use Figure 8.17,b, where the dynamic model of a mechanism with elastic couplings, placed on both parts of the system, with the /292 presence of a clearance in the kinematic pair, is given. The presence of elastic couplings introduces substantial peculiarities into the motion of a vibro-impact system. We shall learn about them using the example of one of such models, which also has an important practical value.

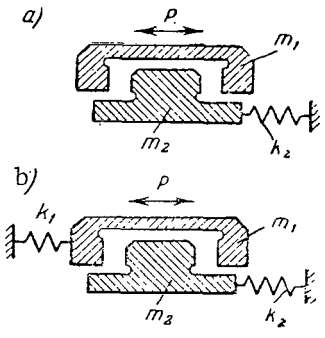

Figure 8.17.

§ 8.5. Elements of the Impact Vibration-Damping Theory. Free Oscillations.

In the preceding chapter we stated that the method of impact vibration damping is widely used even at present in the most diverse branches of technology. The several systems given there, do not exhaust by far the existing, and even less, the possible design solution, enabling us to use this simple and effective method for eliminating the vibration of one-mass and multimass elastic systems, as well as systems with distributed parameters. Here, however, we shall limit ourselves to the examination of one of the simple systems, representing a linear elastic system (m_1, k_1), equipped with vibration dampers of the impact action m_2, the dynamic model of which is given in Figure 8.18. The vibration-damping element is installed with a 2r clearance. With a sufficiently intensive actuation, applied to the elastic system, both parts of the system will move with the collisions. In this case, the forced oscillations of the damped system, will be accompanied by its free harmonic oscillations, arising as the result of collisions. The amplitude of the free motions and their phases may be different after each of the collisions, and then the system's motion will be aperiodical. In other cases, these amplitudes and phases periodically assume the same values, and then the system's motion will be periodical. We shall, as before, look for such periodic motions, which will be characterized by successive alternation of collisions of right and left planes of elements m_1 and m_2. We note, that for the system examined, as well as for the nonlinear element with a clearance, various motions of this /293 type are possible, which differ by the fact that two collisions occur during different numbers 2n + 1 (n = 0, 1, 2, ...) periods of the action of the external force. Let us simplify our problem by limiting ourselves to the investigation of the most effective means of vibration damping, in the process of which, two collisions occur per period of action of the external force (case n = 0).

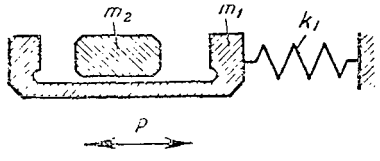

Figure 8.18.

First, let us see, what is the effect of the presence of the vibration-damping element on the system's own properties, in particular on the frequency of its free vibrations. In the case of the absence of the vibration-damping elements this frequency is determined, as usual,

$$\omega_1 = \sqrt{\frac{k_1}{m_1}}.$$

If mass m_2 is installed without a clearance, the frequency of free oscillations will be

$$\Omega_1 = \sqrt{\frac{k_1}{m_1 + m_2}}.$$

In both cases, the elastic system in the process of free oscillations will move harmonically, if we consider that it possesses the property of conservation, i.e, if we do not take into account the loss of energy for overcoming external or internal resistance.

Now, we impart the initial perturbation to the elastic system with a vibration-damper, installed with a clearance, and pose the following two questions. Can this system perform periodic motions of the type that interests us, i.e., such motions, in the process of which in the period of its free oscillations, two collisions occur? If it can, what will be the frequency of such free oscillations with collisions?

Obviously, these questions are sensible only if we consider that the vibro-impact system examined possesses the properties of conservation, For this we must disregard the action of the forces of internal and external friction, and consider the value of the coefficient of restitution in the impact equal to R = 1. Attributing to the system the property of conservation we shall attempt to answer the question posed, for which we shall use the conditions of periodicity, peculiar to the conditions sought. By analogy with (8.3) we can write /294

$$\left.\begin{aligned} &x_1^* = x_c^*, \quad x_2^* = x_c^* + r, \quad \dot{x}_1^* = v_2, \ \dot{x}_2^* = -u \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{when } t = 0, \\ &x_1^* = -x_c^*, \ x_2^* = -(x_c^* + r), \ \dot{x}_1^* = v_1, \ \dot{x}_2^* = -u \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{when } t = \frac{\pi}{\bar{\omega}}. \end{aligned}\right\} \tag{8.35}$$

These conditions differ from conditions (8.3) by the fact that instead of the frequency of the compelling force ω here, a so-far unknown frequency $\bar{\omega}$ of free oscillations with collisions is figuring here; it is also assumed that the vibration-damping element owing to the action of the forces of friction, moves with a constant speed. Moreover, the measured form of record is maintained here.

In the intervals between the collisions, mass m_1 moves harmonically, as the usual linear system:

$$\left.\begin{aligned} &x_1^* = x_{10}^* \cos(\omega_1 t + \psi), \\ &\dot{x}_1^* = -\omega_1 x_{10}^* \sin(\omega_1 t + \psi) \end{aligned}\right\} \quad 0 \leqslant t \leqslant \frac{\pi}{\bar{\omega}}; \tag{8.36}$$

x_{10}^* and ψ are the sought amplitude and phase of the free oscillation.

Mass m_2 moves according to the law

$$x_2^* = (x_c^* + r) - ut. \tag{8.37}$$

Substituting in (8.36) the periodicity conditions, we find that

$$\psi = (1 - \bar{\zeta})\frac{\pi}{2}, \quad \bar{\zeta} = \frac{\omega_1}{\omega},$$
$$v_2 = v_1 = v.$$

As we can see, the velocity of the damped system at the beginning and end of the free motion interval, must be equal. The relationship between the velocities of both masses will be found by applying the pulse theorem (1.11). With R = 1, it yields

$$u = \mu v, \quad \mu = \frac{m_1}{m_2}.$$

Using the values found, we can write the law of motion (8.36) in the following form: /295

$$x_1^* = \frac{u}{\mu\omega_1 \cos\frac{\pi\bar{\zeta}}{2}} \sin\left(\omega_1 t - \frac{\pi\bar{\zeta}}{2}\right).$$

Substituting here and in (8.37) the value $t = \pi/\bar{\omega}$, we get the following two equations:

$$u = \frac{2(x_c^* + r)\omega}{\pi},$$
$$-x_c^* = \frac{u}{\mu\omega_1}\tan\frac{\pi\bar{\zeta}}{2}.$$

These two equations contain three unknown values ω, u, x_c^*. In order to determine them we need a third equation, which we set up in the form of an energy balance, equating the energy of the system at the moment of collision to the energy imparted to the system as the result of the initial perturbation. The full energy of the system at the moment of impact is

$$E_c = \frac{k_1}{2}\left(x_c^{*2} + \frac{1+\mu}{\mu}\frac{u^2}{\omega_1^2}\right).$$

The energy imparted to the system as the result of the initial perturbation, in the general case, may be written as follows:

$$E_0 = \frac{k_1}{2}\left(X_{10}^2 + \frac{\dot{X}_{10}^2}{\omega_1^2} + \frac{\dot{X}_{20}^2}{\mu\omega_1^2}\right),$$

where X_{10} is the deformation of the elastic coupling as the result of the initial displacement, $\dot{X}_{10}$, $\dot{X}_{20}$ are the initial velocities of the system's parts. For the convenience of the further analysis, we express value E_0 with a certain generalized initial displacement of system X_0, the value of which will be selected in such a way that equality

$$E_0 = \frac{k_1}{2} X_0^2$$

will be fulfilled.

Then, equating $E_c = E_0$, we obtain a third equation, connecting the values sought:

$$u = \mu\omega_1 \sqrt{\frac{X_0^2 - x_c^{*2}}{1+\mu}}.$$

This equation reflects the fact that in the process of free oscillations /296 both parts of the system at the moments of impacts move toward one another ($X_0 > x_c^*$).

Now, we can go over to the dimensionless form of recording, presenting the three equations obtained by us in the following form:

$$\lambda = (\sigma + x_c)\frac{2}{\pi}, \quad x_c = -\frac{\pi}{\mu\bar{\zeta}}\tan\frac{\pi\bar{\zeta}}{2}, \quad \lambda = \mu\bar{\zeta}\sqrt{\frac{1-x_c^2}{1+\mu}}.$$

In these equations

$$\lambda = \frac{u}{X_0\bar{\omega}}, \quad x_0 = \frac{x_c^*}{X_0}, \quad \sigma = \frac{r}{X_0}.$$

Solving them jointly, we find that

$$\left.\begin{aligned} \sigma &= \frac{\tan\frac{\pi\bar{\zeta}}{2} + \mu\frac{\pi\bar{\zeta}}{2}}{\sqrt{\tan^2\frac{\pi\bar{\zeta}}{2} + (1+\mu)}}, \quad \lambda = \frac{\mu\bar{\zeta}}{\sqrt{\tan^2\frac{\pi\bar{\zeta}}{2} + (1+\mu)}}, \\ x_c &= -\frac{\tan\frac{\pi\bar{\zeta}}{2}}{\sqrt{\tan^2\frac{\pi\bar{\zeta}}{2} + (1+\mu)}}. \end{aligned}\right\} \qquad (8.38)$$

The first of these equations represents an equation of the frequency characteristic of the system, connecting the value $\bar{\omega}$ sought with the system's parameters ω_1, μ, and r, as well as with value X_0. Here we have again encountered one of the typical properties of nonlinear systems. As we know, the free oscillation frequency ω_1 of the usual linear system does not depend on the amplitude of the initial perturbation. At the same time frequency $\bar{\omega}$, peculiar to the vibro-impact system examined, depends substantially on the relative clearance σ and consequently on the value of the initial pertubation.

The corresponding characteristics calculated for several values of μ are given in Figure 8.19. The calculation of the characteristics in this example is limited by the time interval $0 < \bar{\zeta} < 2$, inasmuch as the collision period

$$T_c = \frac{\pi}{\bar{\omega}} = \frac{\bar{\zeta}\pi}{\omega_1}$$

must be less than the free oscillation period, /297 which is equal to

$$T_1 = \frac{2\pi}{\omega_1}.$$

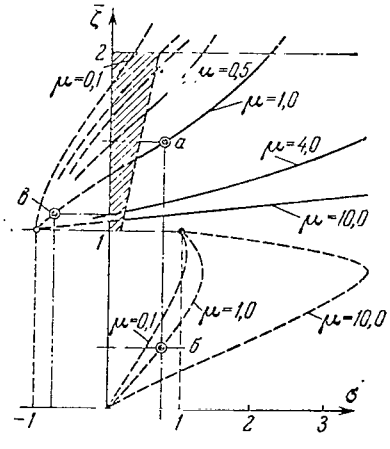

Figure 8.19.

Only under this condition, at least two collisions occur in the period of the movement of the mass, as it was specified earlier. Using the same method it is not hard to examine the periodic motions in the process of which two collisions occur during three, five, ..., (2n + 1) periods of free oscillations of mass m_1. To these conditions will correspond the regions of values $\bar{\zeta} > 2$.

As indicated by the characteristics in Figure 8.19, ambiguous solutions are peculiar to free oscillations of the vibro-impact system. To the same values of parameters of the system and motion correspond two different conditions of free oscillations of a system and two different values of $\bar{\omega}$ respectively. In Figure 8.19 as an example, we have marked two points, a and b, for which $\mu = 1$ and $\sigma = 0.75$. The corresponding laws of the motion of both parts of the system are plotted in Figure 8.20, a and b. According to the third equation of (8.38), value $x_c > 0$ always corresponds to values $\bar{\zeta} > 1$, and values $x_c < 0$ correspond to values $\bar{\zeta} < 1$. In other words, in the first case, masses m_1 and m_2 at the moment of collision are displaced in the same direction from the central position, and in the second case, in different directions.

Plotting of the frequency characteristic may be formally generalized also for the case of the "negative clearances", as it was shown in Figure 8.19,

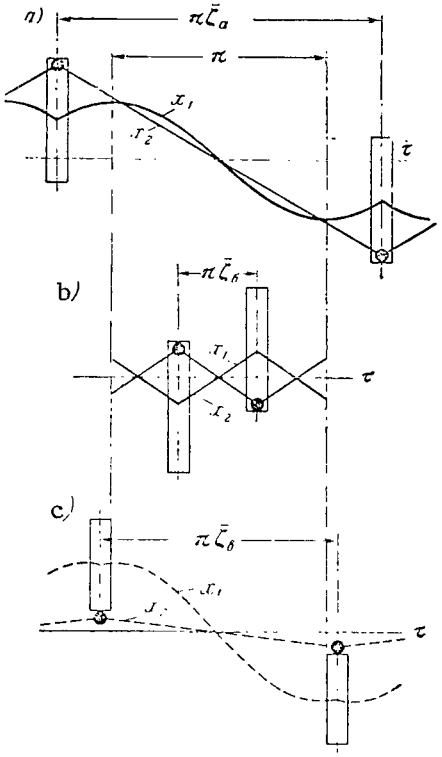

Figure 8.20.

where point b is marked ($\mu = 1$, $\sigma = -0.75$). In Figure 8.20, c, we show the motion corresponding to such a "negative" clearance. This instance has a quite definite physical sense. However, it corresponds to the system, the dynamic model of which differs from the one examined by us, and therefore we shall not analyse it here. /298

Having established the conditions with which, during the period of motion of a system, not less than two collisions occur, we determine the minimum values of the clearance, with which there would be not more than two collisions. For this we must make use of the inequality proceeding from the condition analogous to (8.11), which excludes the possibility of additional collisions, in the interval of motion examined. /299

The results of the corresponding calculations are given in Figure 8.19, in the form of a broken line, dividing the regions of value σ in the interval of $1 < \bar{\zeta} < 2$ into two sectors. The shaded sector corresponds to those values of the system's parameters, with which the periodicity of its motion is disrupted as the result of additional collisions.

The laws of motion of the system being recorded in the dimensionless form, have the following appearance:

$$x_1 = \frac{\lambda}{\mu\bar{\zeta}\cos\frac{\pi\bar{\zeta}}{2}}\sin\bar{\zeta}\left(\tau-\frac{\pi}{2}\right),$$

$$x_2 = (x_c+\sigma)-\lambda\tau, \quad 0\leqslant\tau\leqslant\pi.$$

With these relationships we built the curves in Figure 8.20. Substituting the values for λ here, according to (8.38), we find the amplitude of the oscillations of the system being damped

$$x_{10} = \frac{x_{10}^*}{X_0} = \frac{1}{\sqrt{\sin^2\frac{\pi\bar{\zeta}}{2}+(1+\mu)\cos^2\frac{\pi\bar{\zeta}}{2}}}.$$

As we can see, this amplitude is always $x_{10} < 1$. Therefore, even under

the condition of full conservation, the amplitude of the oscillations of the system equipped with a vibration damper is always less than the amplitude of oscillations of a free system.

This conclusion cannot be verified experimentally since it does not appear possible to build a model of a vibro-impact system, for which the restitution coefficient during an impact would be R = 1. However, experiments show that for real systems, the presence of a vibration damper always improves /300 the damping properties of a system. Such experiments were performed by means of a special model of an elastic system with a vibration damper (Figure 8.21). It represents two elastic rods 3 (their total rigidity is equal to k_1), fixed into a common stand 4 and connected with a hollow body 1. Into the body two stops 2 are screwed in, forming with it, mass m_1 of the system damped. Into the hollow formed by the body and stops, a damping element is placed, consisting of a ball the mass of which is m_2. Wire tensometers 5, glued onto the rods and internal surfaces of the stops, make it possible to register the /301 oscillations of mass m_1 and the moments of its collisions with the damping elements. In Figure 8.22 we give three oscillograms, recorded for various sizes of the clearance. In different series of experiments the amplitude of the initial perturbation was changed in such a way that the initial clearance σ assumed the most diverse values. In all the instances the substantial difference in the speed of attenuation of the oscillations of a system without a clearance (Figure 8.22,a) and with a clearance (8.22,b,c) was evident. A series of experiments in connection with the investigation of the forced oscillations of a system with a vibration damper was also carried out with this model (see below)

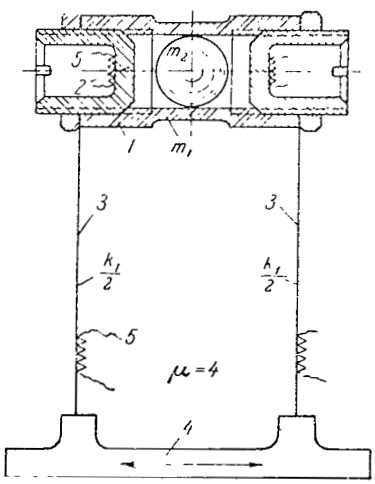

Figure 8.21.

In the conclusion of this paragraph we shall examine two extreme cases. One of them arises when $\mu \to 0$ ($m_2 \to \infty$). At its extreme this instance corresponds to the movement of an elastic system (k_1, m_1), accompanied by its impacts against the immobile limiter (Figure 8.23). Here, according to (8.38),

$$\sigma = \sin \frac{\pi}{2} \bar{\zeta} \leqslant 1,$$
$$\lambda = 0,$$
$$x_c = -\sigma.$$

As we can see, when $\sigma > 1$ ($X_0 < r$) there are no collisions, with $\sigma = 1$ there are periodic contacts (but not collisions!) of both masses; here, naturally

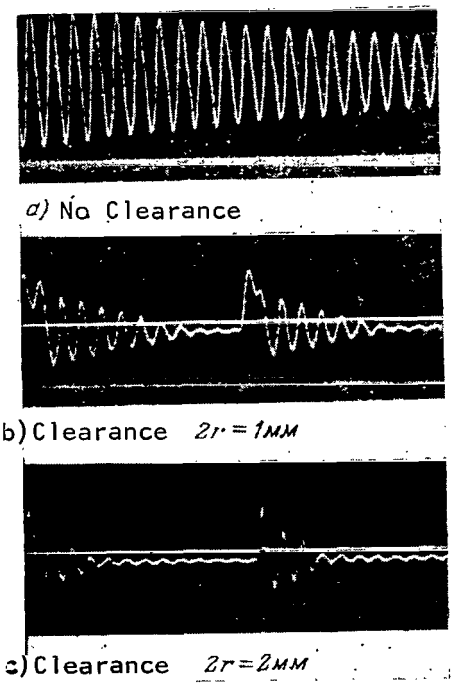

Figure 8.22.

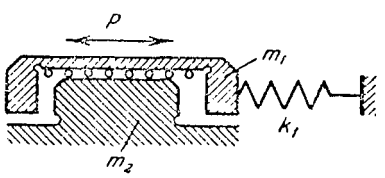

Figure 8.23.

$\bar{\omega} = \omega_1$; with the further increase of the amplitude of the initial perturbation ($X_0 >> r$) frequency $\bar{\omega}$ grows continuously.

The other extreme case takes place when $\mu \to \infty$ ($m_2 \to 0$). Having divided the first and third equations in (8.38) one by the other, we find

$$\left(\frac{\sigma}{x_c} + 1\right) \tan \frac{\pi}{2} \bar{\zeta} = -\mu \frac{\pi}{2} \bar{\zeta}. \qquad (8.39)$$

From here it follows that when $\mu \to \infty$ value $\bar{\zeta} \to 1$; in other words, the obvious result, /302 that with the decrease of the mass of the damping element value $\bar{\omega} \to \omega_1$, is corroborated.

§ 8.6. Elements of the Theory of Impact Vibration Damping. Forced Oscillations.

Let us assume that under the action of a harmonic force $P = P_0 \cos \omega t$ the periodic motion of an elastic system with a vibration damper has been established, which occurs with a frequency ω and satisfies the conditions of periodicity (8.35), when values $\bar{\omega}$ in them are replaced with ω. Now we shall abandon the assumption that the system is conservative and will consider that the coefficient of restitution may have any value $0 < R \leq 1$.

In the intervals between collisions the movement of the mass satisfies the linear differential equation

$$m_1 \ddot{x}_1^* + k_1 x_1^* = P_0 \cos \omega t.$$

Accordingly, we write the laws of its motion in the following way:

$$\left.\begin{aligned} x_1^* &= x_{10}^* \cos(\omega_1 t + \psi) + x_{20}^* \cos(\omega t + \varphi), \\ \dot{x}_1^* &= -\omega_1 x_{10}^* \sin(\omega_1 t + \psi) - \omega x_{20}^* \sin(\omega t + \varphi). \end{aligned}\right\} \qquad (8.40)$$

Mass m_2 in the intervals between collisions moves with a constant velocity, so that as before

$$x_2^* = (x_c^* + r) - ut. \tag{8.41}$$

Using these laws of motion, the periodicity condition, and the pulse theorem, we get the following dimensionless expressions from the amplitudes and phases of free and forced oscillations respectively:

$$\left.\begin{aligned} x_{10} &= -\frac{\lambda}{\mu}\frac{1}{\zeta\cos\frac{\pi}{2}\zeta}\ ;\ \psi = -\frac{\pi}{2}(1-\zeta),\quad \zeta = \frac{\omega_1}{\omega}\ , \\ x_{20} &= \frac{\zeta^2}{\zeta^2-1}\ ,\quad \sin\varphi = b\lambda\frac{\zeta^2-1}{\zeta^2}\ ,\quad b = \frac{1+\mu}{\mu}\frac{1-R}{1+R}\ . \end{aligned}\right\} \tag{8.42}$$

In these formulas $x_{10} = x_{10}^*/a$, $x_{20} = x_{20}^*/a$, $\lambda = u/\omega a$, $a = \dfrac{P_0}{m_1\omega_1^2} = \dfrac{P_0}{k_1}$ is the value of static deformation under the effect of force P_0.

The laws of motion of both parts of the system assume here the following appearance: /303

$$x_1 = \frac{\lambda}{\mu\zeta\cos\frac{\pi\zeta}{2}}\sin\zeta\left(\tau-\frac{\pi}{2}\right)+\frac{\zeta^2}{\zeta^2-1}\cos(\tau+\varphi)$$

$$x_2 = (x_c+\sigma)-\lambda\tau,\qquad 0\leqslant\tau\leqslant\pi.$$

These equations fully determine the motion of the system damped, its form and amplitude as the function of parameters of the system and actuation.

A substantial feature of any vibro-impact system with elastic couplings consists of the fact that the established conditions of its motion are the result of the imposed forced and free oscillations, arising after every collision.

Owing to this feature, the usual method of frequency analysis of established oscillations, which uses amplitude and phase characteristics, is inapplicable here.

In order to characterize more or less fully the movement of an elastic vibro-impact system, we have to use not the amplitude or phase of forced oscillations, but another parameter, more convenient for frequency analysis. As indicated by formulas (8.42), such a parameter is the dimensionless velocity λ of the vibration-damping element. The amplitude of free oscillations and the phase of forced oscillations is expressed through it linearly. The linear functions of this parameter are the speeds of the damped system at the

moments before and after collisions, as well as the coordinates of the collisions:

$$v_1 = \frac{2+\mu-\mu R}{\mu(1+R)}\lambda, \quad v_2 = \frac{(1+R)\lambda-(\mu-R)v_1}{1+\mu},$$
$$x_c = \frac{\pi\lambda}{2} - \sigma, \qquad x_c = \frac{x_c^*}{a}, \quad \sigma = \frac{r}{a}.$$

Value λ, as it will be shown later, determines the dissipative properties, and in the other cases the dynamic accuracy of the system, the effectiveness of its action, etc. Value λ is expressed through the parameters of the system and actuation in the following way: /304

$$\left.\begin{aligned} \lambda &= \frac{A\sigma \pm b\sqrt{B^2(A^2+b^2)-\sigma^2}}{A^2+b^2}, \\ A &= \frac{\pi}{2} + \frac{1}{\mu\zeta}\tan\frac{\pi}{2}\zeta, \quad B = \frac{\zeta^2}{b(\zeta^2-1)}. \end{aligned}\right\} \qquad (8.43)$$

In accordance with the periodicity conditions, we must always select the root of $\lambda > 0$. The periodic conditions of the type examined are possible only under these conditions. However, this condition does not exclude the ambiguity of the solution (see below, Figure 8.24). With the same parameters of actuation and system, two different conditions of its forced oscillations are possible.

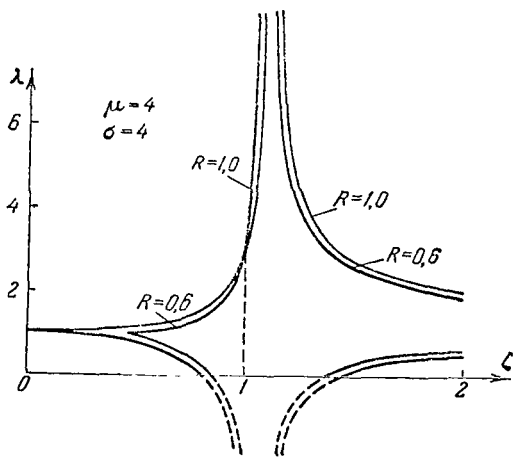

Figure 8.24.

Equation (8.43) to the analysis of which we shall return, may be considered as a frequency characteristic $\lambda = \lambda(\zeta)$ with fixed values of parameters ω_1, μ, σ, and R. First of all we determine the features of this characteristic in the case when the system possesses conservative properties (R = 1). In this case b = 0, and then

$$\lambda = \frac{\sigma \pm \frac{\zeta^2}{\zeta^2-1}}{\frac{\pi}{2} + \frac{\tan\frac{\pi\zeta}{2}}{\mu\zeta}}. \qquad (8.44)$$

We determine the value of λ in the initial point of the characteristic, coinciding with value $\zeta = 0$. It proves equal to

$$\lambda = \frac{2\sigma\mu}{\pi(1+\mu)}.$$

For the second special point of characteristic $\zeta = 1$, corresponding to the resonance conditions of the usual linear system, having opened the indeterminancy, we find

$$\lambda = \pm \frac{\pi\mu}{4}. \tag{8.45}$$

As it was already mentioned, $\lambda > 0$ corresponds to the model examined. Later we shall return to the analysis of the conditions of motion with $\zeta = 1$, and now we shall determine one more point of the characteristic, selecting it in such a way as to satisfy equation /305

$$\tan \frac{\pi\zeta}{2} = -\mu \frac{\pi\zeta}{2}. \tag{8.46}$$

Here, it is found that $A = 0$ and according to (8.44), we get $\lambda = \infty$.

Equation (8.46) takes place when value $\zeta > 1$. Therefore, the point of break of the characteristic is displaced toward a lower frequency then frequency ω_1 of the oscillations of the damped system. Comparing (8.46) with (8.39) we see that when $\sigma = 0$, they coincide with the $\omega = \bar{\omega}$ value.

When $\sigma \neq 0$, the frequency of actuation with which the characteristic suffered a break, was located in the vicinity of frequency $\bar{\omega}$ of the free oscillations of the vibro-impact system.

We should note, that the transcendent equation (8.46) is satisfied with an infinite number of values of ζ. Examining such periodic motions, in the process of which two collisions occur per period of action of the actuating force, we shall limit ourselves to plotting the characteristics $\lambda = \lambda(\zeta)$ on the sector $0 < \zeta < 2$, within the bounds of which all the possible conditions of such types are located. This, of course, does not exclude the possibility of continuation of plotting of the characteristics in the direction of $\zeta > 2$, and examining, if necessary, the other periodic conditions of the movement of the system, in the process of which two collisions occur during three, five, ..., $(2n + 1)$ periods of action of the actuating force. Here, the characteristic

$$\lambda = \lambda(\zeta)$$

will undergo a break every time when ζ assumes the values satisfying equation (8.46).

Now we shall return to equation (8.43), which is correct for any values $0 < R < 1$. The region of actual values of λ here is limited by the inequality

$$\sigma \leq B\sqrt{A^2 + b^2}.$$

By this inequality we determine the extreme point of the characteristic /306

when $\zeta = 0$.

When $\zeta = 1$ formula (8.43) as well as formula (8.44) results in an ambiguity. Having opened this ambiguity we find (see (8.45))

$$\lambda = \pm \frac{\pi\mu}{4}.$$

As we can see, here value λ does not depend on R and σ. Finally, when $A = 0$, i.e., when equality (8.46) is fulfilled, we get

$$\lambda = \pm \frac{\sqrt{b^2B^2 - \sigma^2}}{b}.$$

Owing to the fact that the system now possesses dissipative properties ($R < 1$), the value of λ becomes limited, which indicates the possibility of establishing periodic conditions of motion of the type examined. Here value λ has a real value under the conditions that

$$\sigma < bB.$$

Upon the increase of the relative clearance to

$$\sigma = bB = \frac{\zeta^2}{\zeta^2 - 1}$$

value $\lambda = 0$ and the collisions cease. Mass m_2 may be considered immobile; the damped system performs forced oscillations, periodically touching mass m_2.

Naturally, such a picture proceeds from a purely formal analysis of the relationships obtained. Actually, the random perturbations and forces of friction between the damped system and the vibration-damping element will result in the fact that both parts of the system will in some way interact with one another.

Thus, the analysis of frequency characteristics of parameter λ enables us to obtain a sufficiently full concept of the character and properties of the examined periodic motion of the system. In Figure 8.24 we give two such characteristics, plotted in the $0 < \zeta < 2$ interval.

In concluding this paragraph we shall return to the analysis of the system's movement when $\zeta = 1$ ($\omega = \omega_1$). The presence of a vibration-damping element imparts damping properties to the system, through which it will perform periodic motions of a limited amplitude. This is clear, at least because of the fact that earlier, in the analysis of the frequency characteristic, we found the finite value of λ, peculiar to a certain periodic motion of the vibro-impact system. However, we cannot find the corresponding law of motion of the system by using solution (8.40), inasmuch as with value $\zeta = 1$, they lose their force. From the theory of linear differential equations we

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know that here, the solution of the differential equation, describing the motion of the damped system, should be sought in the form of

$$x_1^* = x_{10}^* \cos(\omega_1 t + \omega) + x_{20}^* t \sin(\omega_1 t + \varphi).$$

By applying the already known method, i.e., by using the former conditions of periodicity and pulse theorem for finding the amplitude and phase values sought in this solution, we get, as the result of a simple analysis the laws of motion of both parts of the system in the following form: /308

$$\left.\begin{aligned} x_1 &= x_{10}\cos(\tau+\psi) - \frac{\tau}{2}\sin\tau, \\ x_2 &= (x_c+\sigma) - \lambda\tau, \quad 0 \leqslant \tau \leqslant \pi, \\ x_{10} &= \frac{1}{4}\sqrt{\left(\frac{\pi^2\mu}{2}-\sigma\right)^2 + (b\mu-1)^2\pi^2}, \tan\psi = \frac{\pi(b\mu-1)}{\frac{\pi^2\mu}{4}-\sigma}; \\ \lambda &= \pm\frac{\pi\mu}{4}. \end{aligned}\right\} \qquad (8.47)$$

According to relationships (8.47) we plot the curves in Figure 8.25, representing the trajectories and velocities of the damped system (solid lines) and of the vibration-damping element (broken lines).

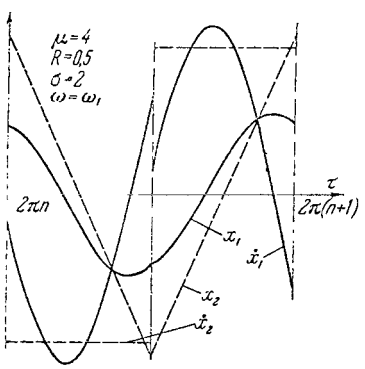

Figure 8.25.

§ 8.7. Damping Properties of the Impact-Action Vibration Damper. Results of the Experiments.

After carrying out the analysis of the periodic motion, it is not difficult to determine the amount of energy dissipated in the process of this motion, and in this way to obtain an evaluation of the dissipative properties acquired by the system equipped with an impact-action vibration damper.

As we know, kinetic energy T, dissipated in the direct central collision of two bodies, is expressed through the velocities of these bodies at the moment prior to the impact, in the following manner:

$$T = \frac{1-R^2}{2}\cdot\frac{m_1 m_2}{m_1+m_2}(v_1-u)^2.$$

As applied to the system examined, this equation, being written in the dimensionless form, acquires the following appearance: /309

$$\xi = \frac{2b}{\mu}\lambda^2,$$

where

$$\xi = \frac{T}{m_1 a^2 \omega^2}$$

characterizes the relative consumption of energy delivered to the damping system from the source of actuation. We consider such periodic conditions, under which two collisions occur, to be a full cycle of the system's motion. Accordingly, the amount of energy lost by the system during one cycle, is equal to

$$\xi_c = \frac{4b}{\mu}\lambda^2. \tag{8.48}$$

Value ξ_c may serve as the quantitative characteristic of the dissipative properties of a system with an impact-action vibration damper. As it should have been expected, the effectiveness of the impact method of vibration damping is fully determined by value λ. Obviously, in slecting the parameters of a vibration damper we should strive to guarantee the maximum value of λ. Along with the mass of the vibration-damping element, a most important parameter, the value of which may be changed within a wide range for this purpose, is the clearance in the coupling of the damping system with the vibration damper. Returning to (8.43), we can determine the value of the clearance with which λ achieves the maximum. Differentiating (8.43) with respect to σ and equating to zero the particular derivative $\frac{\partial \lambda}{\partial \sigma}$ we have

$$\sigma = |AB|.$$

Here the value λ_{max} proves equal to

$$|\lambda_{max}| = |B| = \left|\frac{\zeta^2}{(\zeta^2-1)b}\right|.$$

Substituting this value in (8.48), we have

$$\xi_{c\,max} = \frac{4}{1+\mu}\frac{1+R}{1-R}\left(\frac{\zeta^2}{\zeta^2-1}\right)^2. \tag{8.49}$$

As we can see, with correctly-selected vibration-damper parameters, its characteristic proves quite advantageous from the point of view of the effectiveness of vibration damping. Indeed, the amount of dissipated energy increases sharply upon the approach of the system damped to the resonance state. We should, of course, keep in mind that we can use formula (8.49) only after making sure that the required clearance value has an acceptable dimension. In the work of a system under resonance conditions ($\zeta = 1$) value

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$\lambda = \pi\mu/4$. Consequently

$$\xi_c = \frac{\pi^2}{4}(1+\mu)\frac{1-R}{1+R}. \tag{8.50}$$

In order to make more simple and obvious the evaluation of the effectiveness of the impact-action vibration damper, it is advisable to compare it with the usual linear damper. We know, that if to an elastic system, in addition to a harmonic external force $P_0\cos\omega t$, we apply a linear force of friction $-c\dot{x}_1^*$, the work of this external force during one oscillation cycle is equal to

$$H = \pi P_0 x_1^* \sin\varphi, \tag{8.51}$$

where x_1^* is the amplitude of the established oscillations, ϕ is the phase shift. These values are determined by the following relationships:

$$x_1^* = \frac{P_0}{\sqrt{(k_1 - m_1\omega^2)^2 + c_1^2\omega^2}}, \quad \tan\varphi = \frac{c_1\omega}{k_1 - m_1\omega^2}.$$

Before substituting them in formula (8.51), we express the linear friction coefficient c_i in shares of its critical value c_c. We term critical such a value of the linear friction coefficient, with which the free motion of the elastic system already loses its oscillating character. Returning to Par. 2.1, we can easily show, that such a motion corresponds to the case of multiples of the roots of the characteristic equation (2.3). For the linear oscillation system with friction, examined by us, having the form of

$$m_1\ddot{x}_1^* + c_1\dot{x}_1^* + k_1 x_1^* = 0,$$

the roots of the characteristic equation are determined as follows: /311

$$s_{1,2} = -\frac{c_1}{2m_1} \pm \sqrt{\left(\frac{c_1}{2m_1}\right)^2 - \frac{k_1}{m_1}},$$

and become multiples at the critical friction value:

$$c_c = 2\sqrt{m_1 k_1} = 2m_1\omega_1.$$

Substituting in (8.51) the value $P_0 = m_1 a\omega^2$ and expressing the linear friction coefficient in fractions of its critical value $c_1 = \eta c_c$, we can

determine the work of the external force per cycle of motion in the following manner

$$H = \frac{2\pi m_1 a \omega^2 \eta \zeta}{(\zeta^2 - 1)^2 + 4\eta^2 \zeta^2},$$

or in the dimensionless form:

$$\tau_c = \frac{2\pi \eta \zeta}{(\zeta^2 - 1)^2 + 4\eta^2 \zeta^2}. \tag{8.52}$$

Comparing (8.48) and (8.52) we get

$$\frac{2b}{\mu} \lambda^2 = \frac{\pi \eta \zeta}{(\zeta^2 - 1)^2 + 4\eta^2 \zeta^2}. \tag{8.53}$$

This equation establishes the correspondence between two different methods of dissipation of energy in the process of oscillations. Of course, we should remember that this comparison has a formal character. A system damped by the introduction of a vibration-damping element will not move as does the system with a linear damper. The difference in their motion in the final count will be determined by the fact that in the case of the linear damper, dissipation of energy occurs continuously during the entire period of motion, whereas in the vibration-impact system dissipation of energy occurs in separate pulses.

However, if we regard (8.53) as a common energy relationship, we can use it for the purpose of approximate evaluation of the effectiveness of the impact vibration-damping method. It is especially convenient to do this when we are discussing the use of impact-action vibration dampers for damping the oscillation of multimass system or systems with distributed parameters. With such an approach we may regard the system damped as a linear system, /312
while the effect of the action of the vibration damper on it, can be evaluated by determining the coefficient of the equivalent linear friction

$$c_{eq} = \eta c_c.$$

We perform a similar evaluation for the motion conditions corresponding to the state of the resonance of the linear system. Substituting value $\lambda = \pi\mu/4$ in (8.53) we get with $\zeta = 1$

$$\eta = \frac{2}{\pi(1+\mu)} \frac{1+R}{1-R}.$$

It is not hard to see that even with a relatively small mass of the vibration-damping element we can produce a quite appreciable vibration-damping effect. Thus, for example, when $\mu = 5$, $R = 0.5$, the value of η proves approximately equal to $\eta = 0.32$.

The comparative evaluation given here gives us a visual concept of the wide possibilities of the method of impact vibration damping, the simplicity of realization of which makes it especially convenient for a number of important technical uses.

However, using such an evaluation we should bear in mind that all the results obtained above are correct only under the condition that the movement of the system has a specified periodic character; they lose their significance with such combinations of parameters of the systems and parameters of actuation, when the motion becomes aperiodic or is periodic but differs from the one examined by us. What will be the dissipative properties of the system in its work under these conditions?

An attempt to investigate every one of the possible conditions would be senseless owing to their inexhaustible quantity. An attempt to obtain any integral evaluation of the dissipative properties of a system, which would be correct under any conditions of its motion, would also be unsuccessful. The reasoning that any motion system is accompanied by collisions, and consequently by dissipation of energy, also proves insufficient, for obtaining the required relationships.

In an attempt to answer the question posed, obviously a substantial significance is attained by the experiments, as the result of which it would be possible not merely to verify the deduction formulated on the basis of the analysis of the periodic conditions of motion, but also to get a concept of the system's work under operating conditions differing from those examined in a theoretical investigation. /313

Such experiments were performed with the model given in Figure 8.21. The model is fastened onto the platform of a vibration table (Figure 8.26), which imparts a reciprocating vibration to its base. For this type of actuation, all the earlier-obtained relationships remain in force when we substitute in them

$$a = \frac{\rho_0}{\zeta^2};$$

ρ_0 is the vibration amplitude.

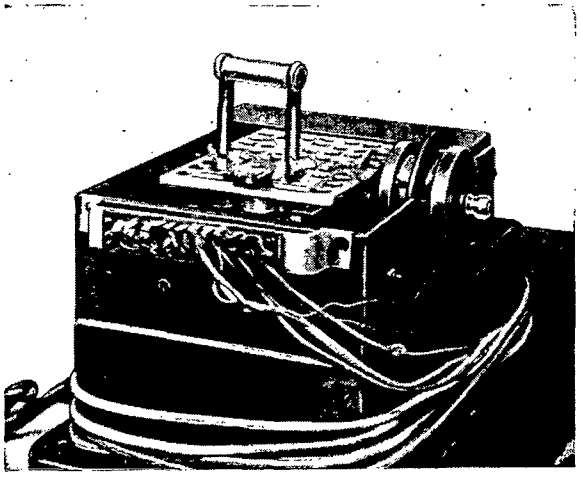


Figure 8.26.

In Figure 8.27 we give the oscillogram of the forced oscillations. The curves on the middle part of the photograph correspond to the motion of the vibration table's platform (curve 1) and the motion of the system damped (curve 2) with respect to the vibrating base. The splashes on the upper and lower straight

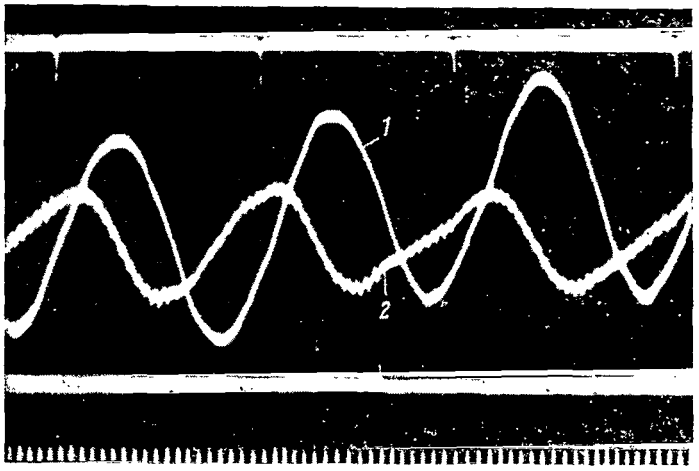
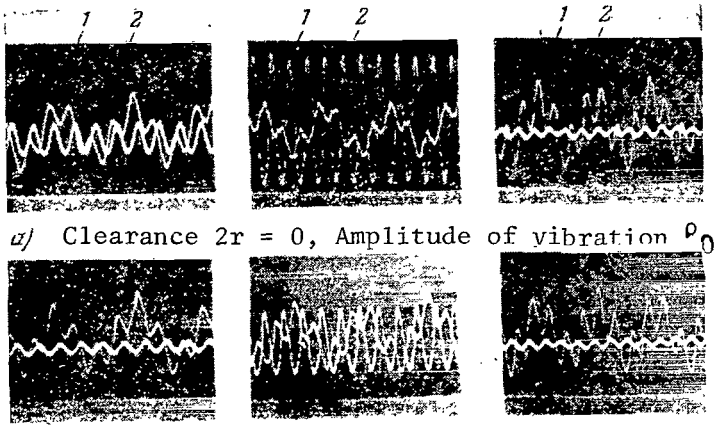


Figure 8.27.



a/ Clearance 2r = 0, Amplitude of vibration ρ_0

b) Clearance 2r - 1mm; Amplitude of vibration, ρ_0

Figure 8.28.

correspond to the moment of collision of the damped system with the vibration damper. As we can see, the motion of the system has a stable periodic character, and in the course of the period of forced oscillations two collisions occur. A certain asymmetry of motion may be explained by the difference of the values of the coefficient of testitution upon the collisions of the vibration-damping element with the left and right stops. /314

In Figure 8.28 we give two series of oscillograms of forced oscillations of the damped system with a fully selected clearance (Figure 8.28) and with a clearance of 2r = 1 mm (Figure 8.28, b). In both instances the amplitude of the vibrations table was similar, while in the three pictures of each series of experiments we recorded the conditions corresponding to the maximum amplitude of oscillations (in the center), as well as the conditions with the actuation frequencies smaller (left) and greater (right) than the first. The picture in the center, in Figure 8.28 a, represents the resonance conditions of the usual linear system (2r = 0). It is not hard to see that upon the introduction of clearance 2r = 1 mm (Figure 8.28, b) the amplitude of the forced oscillations decreases, especially under the conditions close to the resonance conditions. With a further increase of the clearance to the value of 2r = 2 mm, the resonance phenomena have actually disappeared completely. This is shown in Figure 8.29, in which a number of oscillations is given, taken with successive increase of the actuation amplitudes and clearance sizes. All these pictures are made under conditions corresponding to the maximum oscillation amplitude. The first two pictures (a and b) represent the repetition of the pictures given in the /315

preceding diagram. As it had been already mentioned, upon the increase of the clearance to 2r - 2 mm (picture c) the resonance phenomena have practically disappeared. Then the amplitude of the table's vibration was doubled (picture d). In this case the pattern of the phenomenon has been entirely repeated, upon the decrease of the clearance (picture e) the amplitude of the clearance to 2r = 4 mm the resonance phenomena disappeared again. The last three pictures (f, g, and h) are taken when the amplitude of the vibration was increased to $3\rho_0$. And again, the increase of the gap to the maximum for this model value of 2r = 7.6 mm, resulted in a sharp decrease of the oscillation amplitude. /317

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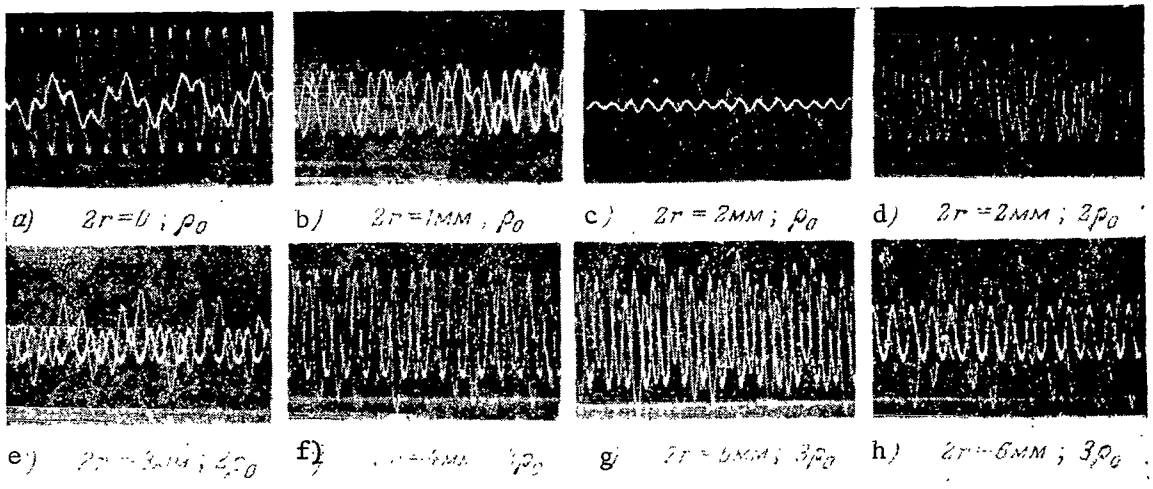

Figure 8.29.

In the experiments the frequencies and amplitudes of the actuations and sizes of the clearances varied within wide ranges. Naturally, in this case there were the most widely different periodic and aperiodic systems of motion. In all the instances the dynamic and dissipative properties acquired by the system with the impact-action vibration damper, resulted in the same effect, a substantial decrease of the maximum oscillation amplitudes in all the conditions of its motion.

CHAPTER 9

ASYMMETRICAL VIBRO-IMPACT SYSTEMS

§ 9.1. N.N. Andreyev's "Little Hammer". /318

We call asymmetric such vibration-impact systems, the periodic motions of which are accompanied by one-sided collisions of their components. A typical example of such a system is the mechanism of the amplitude-meter, shown in Figure 7.10 and given the descriptive name of N.N. Andreyev's "little hammer" by L.I. Mandel'shtam. Formerly it was stated that we are led to the investigation of a perfectly analogous dynamic model by a series of important engineering problems. For us this model is of interest also because it contains two masses and two elastic couplings (see Figure 9.1), and consequently represents a complete two-mass vibro-impact system. We shall devote an appreciable part of the present chapter to the presentation of the methods of calculation of its dynamics and stability. For this, in order to make it easier for the reader to understand this method, we shall give the most important intermediate computations and the dimensional shape of the recording in the initial stage of the presentation.

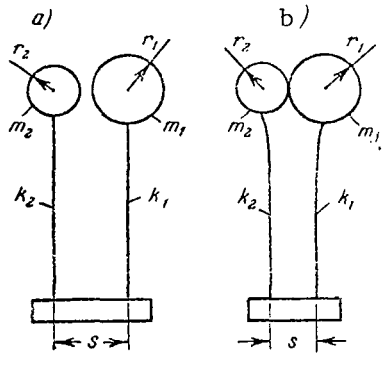

Figure 9.1.

In Figure 9.1 we give two possible variations of installation of both parts of the system: with $r_1 + r_2 < s$ the system is installed with a clearance, if $r_1 + r_2 > s$, we /319 shall say that the system is installed with an initial tight fit (or simply, with a tightfit). We select a count system common for both parts of the system, for any way they are arranged. It is convenient to combine the beginning of the count with a certain point characterizing the position of static equilibrium, which will be established depending on the rigidity values k_1 and k_2 of the elastic couplings and value

$$\delta = (r_1 + r_2) - s$$

of tightness ($\delta > 0$) or clearance ($\delta < 0$). In order to find the position of static equilibrium, we use the condition of equilibrium

$$\frac{d\Pi}{dx^*} = 0.$$

In calculating the potential energy of a system installed with a tight fit, the source of the coordinates will be made to coincide with point 0_1, which

corresponds to the free position of mass m_1, as it is shown in Figure 9.2, in which we portray the characteristics of both elastic couplings and mark the coordinate x_0^* of the position of the static equilibrium of the system. (Point 0_2 corresponds to the free position of mass m_2). Then the expression of the potential energy of the system in the function of the value of displacement x will be written in the following way:

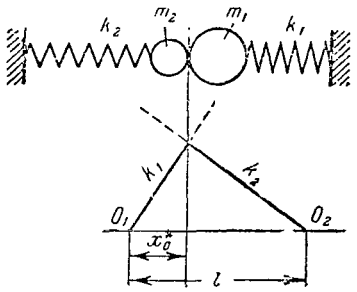

Figure 9.2.

$$\Pi = \frac{k_1 x^{*2}}{2} + \frac{k_2 (x^* - l)^2}{2}.$$

Differentiating, we find coordinate x_0^*, characterizing the position of the static equilibrium:

$$x_0^* = \frac{l}{1+\varkappa}, \quad \varkappa = \frac{k_1}{k_2}. \tag{9.1}$$

The point with coordinate x_0^* will be selected as the start of the count, also in the case when the system is installed with a clearance. In this case we shall have $x_0^* < 0$.

When there is a clearance, both parts of the system, being taken out of the position of equilibrium, may oscillate independently of one another with the frequencies equal to $\omega_1^2 = k_1/m_1$ and $\omega_2^2 = k_2/m_2$ respectively. With a sufficiently large initial perturbation, their motion will be accompanied by collisions. /320

When a system is installed with a tight fit, both masses, being taken out of the position of equilibrium, may oscillate jointly. Here, the movement of the system as a whole, will satisfy the usual linear differential equation; it will perform harmonic oscillations with a frequency of

$$\omega_0^2 = \frac{k_1 + k_2}{m_1 + m_2},$$

the value of which will depend on the value of the initial perturbation. Similar linear conditions of motion of the system examined may be established also in case a harmonic actuation will be applied to one of its parts. However, with a definite value of the initial perturbation, or with specific relationships between the parameters of the system and the parameters of the periodic actuation, the force of the initial tight fit proves insufficient for maintaining contact between both parts of the system. The possibility of "disruptions" of the system, of the formation of nonlinear, with breaks and collisions, conditions of its motion, appears. In many practically important instances, the determination of the condition of origination of such conditions, is of substantial interest. Let us determine this condition first for

the case when the initial perturbation is imparted to the system.

As the result of the initial perturbation, the system, up to the appearance of a break, moves harmoniously

$$\left.\begin{aligned} x^* &= X_b \cos(\omega_0 t + \psi_b), \\ X_b &= \sqrt{x_b^{*2} + \left(\frac{\dot{x}_b^*}{\omega_0}\right)^2}; \quad \tan\psi_b = \frac{\dot{x}_b^*}{x_b^*\omega_0}, \end{aligned}\right\} \tag{9.2}$$

where x_b^* and $\dot{x}_b^*$ are parameters of the initial perturbation.

We determine the condition of origination of a break by calculating the value of the original deviation X_b, with which the force of contact Q of both /321 parts of the system periodically turns to zero.

For this we shall write the equation of motion of one of the masses, for example m_1, in the following form:

$$m_1\ddot{x}_1^* = -(x_0^* + x_1^*)k_1 + Q. \tag{9.3}$$

Substituting in this equation value x_0^* according to (9.1) and x_1^* and $\ddot{x}_1^*$ according to (9.2) and assuming that Q = 0, we find that

$$(k_1 - m_1\omega_0^2) X_b \cos(\omega_0 t + \psi_b) = -\frac{k_1 l}{1+\varkappa}. \tag{9.4}$$

Reducing both parts of this equation by value k_1 and keeping in mind that $k_1/m_1 = \omega_1^2$, we get

$$\cos(\omega_0 t + \psi_b) = \frac{l}{X_b(1+\varkappa)\left[1 - \left(\frac{\omega_0}{\omega_1}\right)^2\right]}.$$

This equation permits us to determine the moment of time, when Q = 0 condition is fulfilled. Consequently, the condition of the origination of the break in which we are interested may be found from the examination of the extreme value of the following inequality:

$$|\cos(\omega_0 t + \psi_b)| \leqslant 1, \tag{9.5}$$

which will be fulfilled if the initial perturbation will be

$$X_b \geqslant \left| \frac{l}{(1+\varkappa)\left[1 - \left(\frac{\omega_0}{\omega_1}\right)^2\right]} \right|.$$

Keeping in mind that

$$\left(\frac{\omega_0}{\omega_1}\right)^2 = \frac{\mu(1+\varkappa)}{\varkappa(1+\mu)}, \quad \mu = \frac{m_1}{m_2},$$

we shall find the following condition of the break, upon the fulfillment of which the force of contact changes its sign or at least turns to zero during the oscillation period:

$$X_b = \left| \frac{l\varkappa(1+\mu)}{(1+\varkappa)(\varkappa-\mu)} \right|. \tag{9.6}$$

We note that with the equality of the free oscillation frequencies of individual parts of system ($\omega_1 = \omega_2$), which corresponds to equation $\kappa = \mu$, its break is impossible under no matter how big an initial perturbation if it is imparted simultaneously to both parts of the system. In the case of the absence of a tight fit ($l = 0$) the rupture of the system occurs with a quite insignificant perturbation, of course under the condition that $\omega_1 \neq \omega_2$. /322

And now we will look at the appearance of the condition of origination of a rupture in the case when a harmonic force $P_0 \cos\omega t$ acts on one of the parts of the system.

The combined steady movement of both parts of the system under the effect of the action of this force satisfies the equation

$$x^* = \frac{P_0 \cos\omega t}{(k_1+k_2)\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]}. \tag{9.7}$$

Let an external force be applied to m_2. We shall examine the movement of mass m_1 using equation (9.3). Substituting x^* and $\ddot{x}^*$ in it according to (9.7), we shall find by analogy with (9.6) the condition of origination of a rupture in the following form:

$$P_0 \geqslant \left| \frac{k_1 l(1+\mu)\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]}{\varkappa-\mu} \right|. \tag{9.8}$$

This condition enables us to determine the force of the tight fit which assures the power closing of the kinematic chain of the mechanism under the prescribed parameters of the system and external perturbation.

§ 9.2. Free Oscillations with Collisions.

We have now seen that in the presence of a tight fit the two-mass system as the result of an intial perturbation may perform harmonic oscillations under the condition that the perturbation amplitude will be smaller than a certain value, which is easily determined from equation (9.6). If, however, the value of the perturbation will exceed its extreme value, or if the system is installed with a clearance, then its motions will be accompanied by ruptures and collisions. Inasmuch as in a real system collisions are always accompanied by scattering of energy, the vibro-impact motion, after a certain time, will be converted again into the usual combined oscillating motion of both systems. /323
In the future this oscillation motion will also die down under the effect of the forces of internal friction. However, at this time we shall examine an idealized system, considering that both colliding bodies are ideally elastic (R = 1) and that the system is free of the forces of external and internal friction.

Applicable to such an idealized conservative system we shall find out if it can perform periodic motions of the vibro-impact type, and try to establish the substantial features of these conditions of motion. The results of such an analysis of free oscillations of a system with periodic collisions in the future will help us clarify the picture of its forced motion.

Applicably to the case examined, we shall write the periodicity conditions as follows:

$$\left.\begin{aligned} x_1^* = x_2^* = x_c^*,\ \dot{x}_1^* = v_2,\ \dot{x}_2^* = u_2 \text{ when } t = 0, \\ x_1^* = x_2^* = x_c^*,\ \dot{x}_1^* = v_1,\ \dot{x}_2^* = u_1 \text{ when } t = \frac{2\pi}{\bar{\omega}}. \end{aligned}\right\} \tag{9.9}$$

Here x_c^* is the collision coordinate v_2, u_2 and v_1, u_1 are the velocities of both parts of the system before and after collision respectively, $\bar{\omega}$ is -the collision frequency.

To make the conditions more definite we assume that the system is installed with a certain tight fit. In the interval between two successive collisions the movements of masses m_1 and m_2 satisfy the following equations:

$$\ddot{x}_1^* + \omega_1^2 x_1^* = -\frac{\omega_1^2 l}{1+\varkappa},$$

$$\ddot{x}_2^* + \omega_2^2 x_2^* = \frac{\omega_2^2 \varkappa l}{1+\varkappa}.$$

The solutions of these equations will be written as follows:

$$\left.\begin{aligned} x_1^* = x_{11}^* \cos(\omega_1 t + \psi_1) - \frac{l}{1+\varkappa}, \\ x_2^* = x_{21}^* \cos(\omega_2 t + \psi_2) + \frac{\varkappa l}{1+\varkappa}. \end{aligned}\right\} \tag{9.10}$$

Differentiating them we find the following equations for masses m_1 and m_2: /324

$$\left.\begin{aligned}\dot{x}_1^* &= -\omega_1 x_{11}\sin(\omega_1 t+\psi_1),\\ \dot{x}_2^* &= -\omega_2 x_{21}\sin(\omega_2 t+\psi_2).\end{aligned}\right\} \tag{9.11}$$

We determine values x_{11}^*, x_{21}^*, ψ_1, and ψ_2 using the periodicity conditions of (9.9). Substituting them in the first of the equations in (9.10), we get for mass m_1, when $t = 0$ and $t = 2\pi/\bar{\omega}$:

$$\left.\begin{aligned}x_c^* &= x_{11}^*\cos\psi_1-\frac{l}{1+\varkappa},\\ x_c^* &= x_{11}^*\cos(2\pi\bar{\zeta}_1+\psi_1)-\frac{l}{1+\varkappa},\quad \bar{\zeta}_1=\frac{\omega_1}{\bar{\omega}}.\end{aligned}\right\} \tag{9.12}$$

Solving these equations jointly we find the value of phase ψ_1:

$$\psi_1 = -\pi\bar{\zeta}_1+\pi n \qquad (n=0,\ 1,\ 2,\ \ldots).$$

Accordingly, for the movement of mass m_2 we find

$$\left.\begin{aligned}x_c^* &= x_{21}^*\cos\psi_2+\frac{\varkappa l}{1+\varkappa},\\ x_c^* &= x_{21}^*\cos(2\pi\bar{\zeta}_2+\psi_2)+\frac{\varkappa l}{1+\varkappa},\end{aligned}\right\} \tag{9.13}$$

from which

$$\psi_2 = -\pi\bar{\zeta}_2+\pi n \qquad \left(\bar{\zeta}_2=\frac{\omega_2}{\bar{\omega}}\right).$$

The values of phases ψ_1 and ψ_2 are not determined unambiguously. However, somewhat later we shall show that we may limit ourselves to accounting for only the principal root of the initial transcendent equation.

Substituting the periodicity conditions in the first of equation (9.11), we find that with an allowance for the arbitrary value of n

$$\begin{aligned}v_2 &= -(-1)^{n+1}\omega_1 x_{11}^*\sin\pi\bar{\zeta}_1 \quad \text{when} \quad t=0,\\ v_1 &= (-1)^{n+1}\omega_1 x_{11}^*\sin\pi\bar{\zeta}_1 \quad \text{when} \quad t=\frac{2\pi}{\bar{\omega}}.\end{aligned}$$

From this it follows that the periodic motion may take place only under the condition that the velocities of mass m_1, before and after the impact, are /325

equal in size and opposite in sign:

$$v_2 = -v_1.$$

Amplitude x_{11}^* is determined as follows:

$$x_{11}^* = \frac{(-1)^{n+1} v_1}{\omega_1 \sin \pi\bar{\zeta}_1}.$$

Precisely in the same way, for mass m_2 we obtain

$$u_2 = -u_1; \quad x_{12}^* = \frac{(-1)^{n+1} u_1}{\omega_2 \sin \pi\bar{\zeta}_2}.$$

According to the pulse theorem, the velocities of both masses before and after collision, are connected by the following relationships (R = 1):

$$u_2 = \frac{(1-\mu) u_1 + 2\mu v_1}{1+\mu}; \qquad v_2 = \frac{2u_1 + (\mu-1) v_1}{1+\mu}.$$

Taking into consideration that $u_2 = -u_1$ we get

$$v_1 = -\frac{u_1}{\mu}$$

and, consequently, the velocities of masses m_1 and m_2 before the impact have different signs.

We substitute in (9.10) the expressions we found for values x_{11}^*, x_{21}^*, ψ_1 and ψ_2 and assume that t = 0, and we shall get

$$x_c^* = -\frac{l}{1+\varkappa} + u_1 \frac{\cot \pi\bar{\zeta}_1}{\mu\omega_1}, \quad x_c^* = \frac{\varkappa l}{1+\varkappa} - u_1 \frac{\cot \pi\bar{\zeta}_2}{\omega_2}. \tag{9.14}$$

As we can see these two equations connecting the three unknown values: x_c^*, u_1, $\bar{\zeta}_1$ (or $\bar{\zeta}_2 = \zeta_1 \sqrt{\kappa/\mu}$), do not depend on the value of n. The third equation, required for determining the parameters of periodic motion, we shall set up in the form of the energy balance.

Let us make use of the circumstance that owing to the conservative nature of the system, energy E_c which it possesses at the moments of collisions, is equal to the initial reserve of energy E_0, which it possesses as the result of deformation of elastic couplings, in the position of static equilibrium, and the initial perturbation imparted to it.

The energy of the system at the moments of collisions is /326

$$E_c = \Pi_c + T_c.$$

The potential energy of system Π_c at the moments of impacts in accordance with Figure 9.3 is equal to

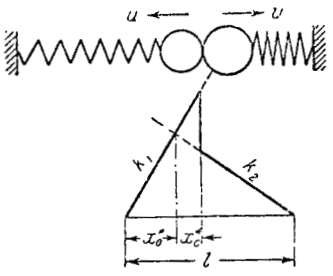

Figure 9.3.

$$\Pi_c = \Pi_{c1} + \Pi_{c2} = \frac{k_1 [l/(1+\varkappa) + x_c^*]^2}{2} + \frac{k_2 [\varkappa l/(1+\varkappa) - x_c^*]^2}{2},$$

from which

$$\Pi_c = \frac{k_2 [x_c^{*2} (1+\varkappa) + \varkappa l^2/(1+\varkappa)]}{2}.$$

Potential energy is a quadratic function of values l and x_c^* and, consequently, does not depend on their signs; expression Π_c remains the same independently of the fact whether the system is installed with a tight fit or a clearance.

The kinetic energy T_c at the moment of collision is equal to

$$T_c = \frac{m_2 u_1^2 (1+\mu)}{2\mu}.$$

The kinetic energy of a system is the quadratic function of velocity and also remains the same when the system is installed according to Figure 9.1,a or 9.1,b.

We now write the energy balance equation

$$E_0 = E_c,$$

from which

$$E_0 = k_2 \frac{[x_c^{*2}(1+\varkappa) + \varkappa l^2/(1+\varkappa)]}{2} + \frac{m_2 u_1^2 (1+\mu)}{2\mu}.$$

We divide both sides of this equation by $m_2\omega_2^2$ and designate

$$\frac{E_0}{m_2\omega_2^2} = X_0$$

X_0 is the reduced amplitude of a certain perturbation, characterizing the /327 initial reserve of energy of the system,

$$X_0 = \frac{[x_c^{*2}(1+\varkappa)+\varkappa l^2/(1+\varkappa)]}{2} + \frac{u_1^2(1+\mu)}{2\mu\omega_2^2}. \qquad (9.15)$$

In this way we have obtained the third equation, connecting two of the three values sought: x_1^*, u_1 and $\bar{\zeta}_1$.

In order to make it unnecessary to prescribe the numerical values of the initial perturbation, we shall now use the dimensionless form of recording, for which we introduce the following designations:

$$X_c = \frac{X_c^*}{X_0}, \quad L = \frac{l}{X_0}, \quad \lambda = \frac{u_1}{\omega X_0}.$$

Now equations (9.14) and (9.15) assume the following form:

$$\left.\begin{aligned} x_c &= \lambda \frac{\cot \pi\bar{\zeta}_1}{\mu\bar{\zeta}_1} - \frac{L}{1+\varkappa}, \\ x_c &= -\lambda \frac{\cot \pi\bar{\zeta}_2}{\bar{\zeta}_2} + \frac{\varkappa L}{1+\varkappa}, \\ \frac{x_c+\varkappa L}{(1+\varkappa)^2} &+ \frac{\lambda(1+\mu)}{\mu(1+\varkappa)\bar{\zeta}_2} = \frac{2}{1+\varkappa}. \end{aligned}\right\} \qquad (9.16)$$

Comparing the first two equations of this system we find

$$\lambda = \frac{L}{\bar{A}}; \quad \bar{A} = \frac{\cot \pi\bar{\zeta}_1}{\mu\bar{\zeta}_1} + \frac{\cot \pi\bar{\zeta}_2}{\bar{\zeta}_2}. \qquad (9.17)$$

The third equation of (9.16) after substitution of values λ and x_c in it will be

$$L = \pm \left| \frac{\sqrt{2}(\cot \pi\bar{\zeta}_1 + \sqrt{\varkappa\mu}\cot \pi\bar{\zeta}_2)}{\sqrt{\varkappa(1+\mu+\cot^2 \pi\bar{\zeta}_1 + \mu\cot^2 \pi\bar{\zeta}_2)}} \right|. \qquad (9.18)$$

This equation represents the frequency characteristic of the system. It enables us, using the prescribed values of the system's parameters κ, μ, and L, to calculate the possible values of $\bar{\zeta}_1$ or $\bar{\zeta}_2$ and then the corresponding frequencies ω_i of the periodic free oscillations with collisions. The plus /328 sign in formula (9.18) corresponds to the case when the system is set up with a tight fit (L > 0), the minus sign, when it is set up with a clearance (L < 0).

Having determined $\bar{\zeta}_1$ or $\bar{\zeta}_2$ we can, using (9.17), find value λ and the collisions coordinates x_c. The laws of motion of both parts of the system in

the interval between collisions are determined by the following relationships:

$$\left.\begin{aligned} x_1 &= -\frac{L}{1+\varkappa} + \frac{\lambda\cos(\tau-\pi\bar{\zeta}_1)}{\mu\bar{\zeta}_1\sin\pi\bar{\zeta}_1}, \\ x_2 &= -\frac{\varkappa L}{1+\varkappa} - \frac{\lambda\cos\left(\sqrt{\frac{\mu}{\varkappa}}\tau-\pi\bar{\zeta}_2\right)}{\bar{\zeta}_2\sin\pi\bar{\zeta}_2} \end{aligned}\right| \qquad (9.19)$$

Upon examination of the free motion conditions of a system we again encounter the properties of nonlinear systems already known to us, namely with ambiguity of the solution. Thus, in Figure 9.4 a, three different laws of free oscillations of the system are presented which correspond to three successive values of roots $\bar{\zeta}_1$ of formula (9.18). However, the/329 number of possible solutions may be sharply reduced. The matter here is that the obvious condition above requiring that no additional collisions which disrupt the periodic conditions should occur in the interval between the calculated collisions. By taking this condition into account we are enabled to separate and exclude from examination the excessive roots of equation (9.18). Thus, in Figure 9.4 a, we can see that the movement with frequency $\bar{\zeta}_{13}$ is accompanied by additional collisions, disrupting the periodic conditions.

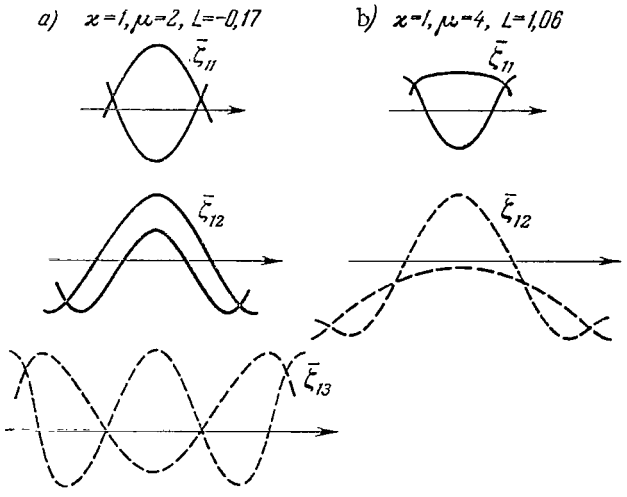

Figure 9.4.

It is not hard to demonstrate that for the system's parameters κ and μ such that $\sqrt{\kappa/\mu} = p/q$, where p and q are whole numbers, the above-mentioned condition is satisfied only by such values of $\bar{\zeta}_1$ which are located in the range $0 < \bar{\zeta}_2 \leq q$. Indeed, under this condition, the laws of motion of system (9.19), as well as the frequency equation (9.18) are periodic functions of $\bar{\zeta}_2$ with a period equal to q. Here, to every value of the root of $\bar{\zeta}_{2i}$ corresponds its own interval of the variation of the dimensionless time τ.

The consecutive values of the roots, for example $\bar{\zeta}_2$, of the frequency equation, may be written as follows:

$$\bar{\zeta}_{11},\ \bar{\zeta}_{12}=\bar{\zeta}_{11}+q,\ \bar{\zeta}_{13}=\bar{\zeta}_{11}+2q,\ \ldots,\ \bar{\zeta}_{1n}=\bar{\zeta}_{11}+(n-1)\,q.$$

Accordingly, we shall determine the intervals of τ variations

$$0\leqslant\tau\leqslant 2\pi\sqrt{\frac{\kappa}{\mu}}\,[\bar{\zeta}_2+(n-1)\,q],\quad n=1,\ 2,\ 3,\ldots$$

According to the periodicity conditions (9.9) at the boundaries of each of these intervals $x_1 = x_2 = x_c$. At the same time each subsequent interval contains within itself the preceding one, and therefore contains the values of τ with which, in the interval examined, additional collisions occur. Because of this, with the indicated relationships of parameters ($\sqrt{\kappa/\mu}$ = p/g we should exclude from the examination all the roots of equation (9.18), lying beyond the limits of $0 \leq \bar{\zeta}_{21} \leq q$. As a descriptive illustration to the above, we may use the curves of the free oscillations of the vibro-impact system examined, presented in Figure 9.4,b. The motion without additional collisions /330 may correspond to only the minimal of all the possible values of $\bar{\zeta}_{1i}$.

We may mention yet another condition, limiting the selection of the possible values of $\bar{\zeta}_1$ or $\bar{\zeta}_2$.

As it was shown above, the velocities of both parts of the system at the moment before the collision have opposite directions. Moreover, with the counting system selected by us, value λ should always be greater than zero. Keeping (9.17) in mind, we find that in this case the roots of equation (9.18) must satisfy the conditions:

$$\left.\begin{aligned}&\cot\pi\bar{\zeta}_1+\sqrt{\kappa\mu}\cot\pi\bar{\zeta}_2>0 \text{ when } L>0,\\ &\cot\pi\bar{\zeta}_1+\sqrt{\kappa\mu}\cot\pi\bar{\zeta}_2<0 \text{ when } L<0.\end{aligned}\right\}\qquad(9.20)$$

In Figure 9.5, as an example we give the frequency characteristic plotted for values κ = 1, μ = 2, and μ = 4. The points mark the values of L and ζ_{in}, for which the laws of motion in Figure 9.4,a and 9.4,b have been plotted.

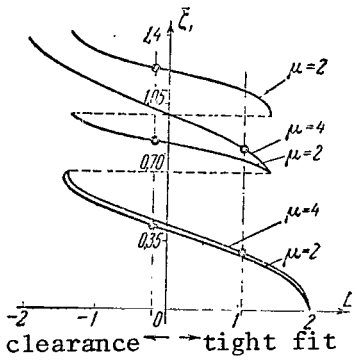

Figure 9.5.

We shall not perform in greater detail here the analysis of the free oscillations of a vibro-impact system. The principal problem connected with the distribution of frequency $\bar{\omega}$ as a function of the parameters of the system has been solved; this will help us understand the picture of the forced oscillations of such a system, in the specific features peculiar to its frequency characteristics, and enable us to evaluate the effect of any parameters of a system on its dynamic properties.

§ 9.3. Forced Oscillations.

Let a harmonic force $P = P_0 \cos\omega t$ (Figure 9.6) be applied to element m_1 of a two-mass elastic system. We assume that as the result of the action of this force a periodic motion has become established, which is accompanied by collisions of both parts of the system with a frequency equal to or a multiple of frequency ω of the forcing power. Let us begin the analysis of such periodic conditions, taking into consideration the dissipation of energy during collisions, considering that the coefficient of restitution R, during the impact, may assume any values within the range of $0 < R \leq 1$.

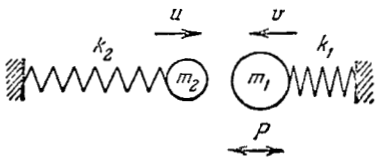

Figure 9.6.

In the intervals between collisions, the movement of masses m_1 and m_2 is described by the equations

$$\left.\begin{aligned} \ddot{x}_1^* + \omega_1^2 x_1^* &= -\frac{l\omega_1^2}{1+\varkappa} + \frac{P_0}{m_1}\cos(\omega t + \varphi), \\ \ddot{x}_2^* + \omega_2^2 x_1^* &= \frac{\varkappa l}{1+\varkappa}\,\omega_2^2, \end{aligned}\right\} \qquad (9.21)$$

ϕ is the phase shift of the external force with respect to the moment of collision. The start of the coordinate count, as it was in the analysis of the free oscillations, will be combined with the position of the static equilibrium. Let us pass over to the dimensionless values, for which we introduce the following designations:

$$x_1^* = \frac{P_0}{m_1\omega_1^2}x_1, \quad x_2^* = \frac{P_0}{m_1\omega_1^2}x_2, \quad \omega t = \tau.$$

Then

$$\frac{dx_1^*}{d\tau} = \frac{P_0}{m_1\omega_1^2}\frac{dx_1}{d\tau}, \quad \frac{d^2x_1^*}{d\tau^2} = \frac{P_0}{m_1\omega_1^2}\frac{d^2x_1}{d\tau^2},$$
$$\frac{dx_2^*}{d\tau} = \frac{P_0}{m_1\omega_1^2}\frac{dx_2}{d\tau}, \quad \frac{d^2x_2^*}{d\tau^2} = \frac{P_0}{m_1\omega_1^2}\frac{d^2x_2}{d\tau^2}.$$

Keeping in mind that

$$\dot{x}_1^* = \frac{dx_1^*}{d\tau}\frac{d\tau}{dt} = \omega\frac{dx_1^*}{d\tau}, \quad \ddot{x}_1^* = \omega^2\frac{d^2x_1^*}{d\tau^2},$$
$$\dot{x}_2^* = \omega\frac{dx_2^*}{d\tau}, \quad \ddot{x}_2^* = \omega^2\frac{d^2x_2^*}{d\tau^2},$$

we shall write the equations of motion (9.21) in the following way:

$$\left.\begin{aligned} \ddot{x}_1 + \zeta_1^2 x_1 &= -\frac{\zeta_1^2 L}{1+\varkappa} + \zeta_1^2 \cos(\tau+\varphi), \\ \ddot{x}_2 + \zeta_2^2 x_2 &= \frac{\zeta_2^2 \varkappa L}{1+\varkappa}. \end{aligned}\right\} \tag{9.22}$$

where

$$\ddot{x}_i = \frac{d^2 x_i}{d\tau^2}; \quad \zeta_i = \frac{\omega_i}{\omega} \ (i = 1, 2), \quad L = \frac{l}{a}, \quad a = \frac{P_0}{m_1 \omega_1^2}.$$

The solution of these equations will be

$$\left.\begin{aligned} x_1 &= C_1 \cos\zeta_1\tau + C_2 \sin\zeta_1\tau - \frac{L}{1+\varkappa} + \frac{\zeta_1^2}{\zeta_2^2 - 1}\cos(\tau+\varphi), \\ x_2 &= C_3 \cos\zeta_2\tau + C_4 \sin\zeta_2\tau + \frac{\varkappa L}{1+\varkappa}. \end{aligned}\right\} \tag{9.23}$$

We write the dimensionless velocities of the masses in the interval between collisions

$$\left.\begin{aligned} \dot{x}_1 &= -C_1\zeta_1 \sin\zeta_1\tau + C_2\zeta_1 \cos\zeta_1\tau - \frac{\zeta_1^2}{\zeta_1^2 - 1}\sin(\tau+\varphi), \\ \dot{x}_2 &= -C_3\zeta_2 \sin\zeta_2\tau + C_4\zeta_2 \cos\zeta_2\tau. \end{aligned}\right\} \tag{9.24}$$

The conditions of periodicity of motions by analogy with (9.9) will be expressed as follows:

$$\left.\begin{aligned} x_1 = x_2 = x_c, \quad \dot{x}_1 = v_2, \quad \dot{x}_2 = \lambda_2 \text{ when } \tau = 0, \\ x_1 = x_2 = x_c, \quad \dot{x}_1 = v_1, \quad \dot{x}_2 = \lambda_1 \text{ when } \tau = 2\pi n \\ (n = 1, 2, 3, \ldots). \end{aligned}\right\} \tag{9.25}$$

Equations (9.23) and (9.25) contain ten unknown values: C_1, C_2, C_3, C_4, ϕ, λ_1, λ_2, v_1, v_2, x_c. Substituting conditions (9.25) in (9.23) and (9.24) we get a system of eight equations for determining these values. We shall obtain two more equations, correlating the velocities of both parts of the system before and after their collisions, using the pulse theorem.

Solving the complete system of equations by the method of consecutive exclusion of unknowns, we can find the values of all the unknowns in the function of the system's parameters. However, we shall do something else. As we did in the preceding chapter, we shall express all the values sought with the dimensionless velocity $\lambda = \lambda_1$, which we shall select as the main characteristic of the system. Here, the record of all the values proves very /333 compact and convenient for further analysis. They are linear functions of parameter λ and are expressed as follows:

$$C_1 = \frac{\cot \pi n \zeta_1}{\mu \zeta_1} \lambda, \quad C_2 = \frac{\lambda}{\mu \zeta_1}, \quad C_3 = \frac{\cot \pi n \zeta_2}{\zeta_2} \lambda,$$
$$C_4 = -\frac{\lambda}{\zeta_2}, \quad \lambda_2 = -\lambda_1 = -\lambda, \quad x_c = \frac{\varkappa L}{1+\varkappa} \frac{\cot \pi n \zeta_2}{\zeta_2}$$
$$v_1 = -\frac{2+\mu-\mu R}{\mu(1+R)} \lambda, \quad v_2 = \frac{2R+\mu+R\mu}{\mu(1+R)} \lambda,$$
$$\cos \varphi = \frac{L - A\lambda}{bB}; \quad \sin \varphi = \frac{\lambda}{B}.$$

Value λ is expressed through the parameters of the system and actuation with the following formula:

$$\lambda = \frac{AL \pm b \sqrt{B^2 (A^2 + b^2) - L^2}}{A^2 + B^2}. \tag{9.26}$$

With respect to its structure this formula coincides with formula (8.43), obtained in the preceding chapter. At the same time the expressions for values A and B in formulas (8.43) and (9.26) differ. Here they have the following appearance:

$$A = \frac{\cot \pi n \zeta_1}{\mu \zeta_1} + \frac{\cot \pi n \zeta_2}{\zeta_2}, \quad B = \frac{\zeta_1^2}{b(\zeta_1^2 - 1)}. \tag{9.27}$$

Just as in formula (8.43), value b is equal to

$$b = \frac{1+\mu}{\mu} \frac{1-R}{1+R}.$$

According to (9.26) value λ is a two-sign function of the parameters, while both of the values may be positive and therefore, satisfy the conditions in (9.25). This circumstance indicates that ambiguous solutions are peculiar to asymmetrical vibro-impact systems, just as to the symmetrical ones.

Turning to the analysis of the frequency characteristic, we shall determine first of all the boundaries of the region of actual solutions. It is written in the following way:

$$B^2 (A^2 + b^2) > L^2,$$

from which

$$L \lessgtr \pm |B \sqrt{A^2 + b^2}|. \tag{9.28}$$

This inequality limits the maximum values of tightness (L > 0) or clearance (L < 0), with which the periodic conditions of the type examined may be established.

With certain particular values of parameters formula (9.26) becomes simplified. Thus, for example, if the system possesses conservative properties (R = 1), then, specifying that b = 0, we find

$$\lambda = \frac{L \pm \frac{\zeta_1^2}{\zeta_1^2 - 1}}{A}$$

Here, the case when A = 0 is especially interesting. By analogy with linear systems it may be regarded as a case of resonance of a vibro-impact system. Value A proves equal to zero, with the actuation frequency ω_1, equal to the frequency of free oscillations $\bar{\omega}$, established in the preceding paragraph. Here, A = $\bar{A}$ (see (9.17)) and corresponds to value L = 0.

Also interesting is the point of the characteristic $\lambda = \lambda(\zeta_1)$, corresponding to value $\zeta_1 = 1$, in other words, corresponding to the resonance of the usual linear system (m_1, k_1). It is not hard to see that with $\zeta_1 \to 1$ values $A \to \infty$, $B \to \infty$ and formula (9.26) do not yield a definite value for λ. It transpires that in this case

$$\lim_{\zeta_1 \to 1} \lambda = \pm \lim_{\zeta_1 \to 1} b \frac{B}{A}.$$

In order to open this indeterminancy we present the last fraction, taking into account that $\zeta_2 = \sqrt{\mu/\kappa}\ \zeta_1$, in the following form:

$$\frac{B}{A} = \frac{\mu\zeta_1^2}{(\zeta_1^2 - 1)\operatorname{cot}\pi n\zeta_1 + \sqrt{\mu\kappa}(\zeta_1^2 - 1)\operatorname{cot}\pi n \sqrt{\frac{\mu}{\kappa}}\zeta_1}. \qquad (9.29)$$

If value $\sqrt{\mu/\kappa}$ is not equal to an integral number or a number of the p/n type, where p is an integral number, then with $\zeta_1 = 1$, the first component in the denominator of the fraction is an indeterminate value of the type of $0 \cdot \infty$, the second value here, being equal to zero.

Applying l'Hopital's rule we find that

$$\lim_{\zeta_1 \to 1} \lambda = \pm \frac{\mu\pi n}{2}.$$

Now, let $\sqrt{\mu/\kappa}$ be equal to a whole number or a number of the type of p/n; /335 then the second component in the denominator of the fraction (9.29) with $\zeta_1 = 1$ also represents an indeterminate value of the $0 \cdot \infty$ type. Opening the indeterminancy of both components of the denominator we find that in this case

$$\lim_{\zeta_1 \to 1} = \pm \lim_{\zeta_1 \to 1} b \frac{B}{A} = \pm \frac{\mu\pi n}{2(1+\kappa)}.$$

When $\zeta_1 = 1$ the particular solution of the first one of the differential equations in (9.22) should be sought in the form of $x_{12}\tau\sin\tau$ (see § 8.6).

We shall now examine the point of the characteristic $\lambda = \lambda(\zeta_1)$, corresponding to the value $\zeta_2 = 1$, or, which is the same, $\zeta_1 = \sqrt{\kappa/\mu}$. If in this case $\kappa \neq \mu$ then, according to (9.29), value $\lambda = 0$. If, however $\kappa = \mu$, then $\zeta_2 = \zeta_1$ and value λ proves equal to

$$\lim_{\zeta_2=\zeta_1=1} \lambda = \pm \frac{\mu\pi n}{1+\varkappa}.$$

Finally, we shall determine the value of λ in the initial point of the characteristic, i.e., with the value of $\zeta_1 = \zeta_2 = 0$. Here, value $A \to \infty$ and according to (9.26)

$$\lim_{\zeta_1=\zeta_2=0} \lambda = 0.$$

Value λ depends substantially on parameter L, characterizing the initial set-up of the system (with a tight fit or a clearance). Under actual conditions this parameter is selected as the regulating one. Formula (9.26) enables us to determine the values of L, which guarantees the extreme values of λ under the prescribed excitation conditions.

Let us find the partial derivative $\frac{\partial\lambda}{\partial L}$ and equate to zero the expression found. It transpires that λ reaches extreme values when

$$L = \pm AB.$$

Here

$$\lambda = \pm B.$$

An approximate appearance of the frequency characteristic $\lambda = \lambda(\zeta_1)$ is presented in Figure 9.7. Here the extreme values of λ take place when $\zeta_1 = \zeta_{1i}$, corresponding to L = 0 (see curve in Figure 9.5). /336

§ 9.4. Stability of Periodic Conditions of Motion.

We shall begin the investigation of the stability of periodic conditions, as usual by setting up the laws of perturbed motion at a certain ν-th interval, limited by the ν-th and $(\nu + 1)$ collision of the system's parts. Introducing the corresponding perturbations into the system of motion (9.23), we get

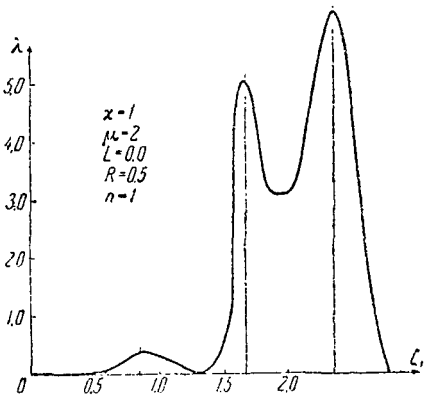

Figure 9.7.

$$\left.\begin{aligned} x_1^{(\nu)} &= (C_1+\varepsilon_{1\nu})\cos\zeta_1\tau+(C_2+\varepsilon_{2\nu})\sin\zeta_1\tau- \\ &\quad -\frac{L}{1+\varkappa}+\frac{\zeta_1^2}{\zeta_1^2-1}\cos(\tau+\varphi+\Delta_{\nu-1}), \\ x_2^{(\nu)} &= (C_3+\varepsilon_{3\nu})\cos\zeta_2\tau+(C_4+\varepsilon_{4\nu})\sin\zeta_2\tau+\frac{\varkappa L}{1+\varkappa}, \end{aligned}\right\} \quad (9.30)$$

where as before $\Delta_{\nu-1} = \sum_0^{\nu-1} \delta_i$ is the accumulated perturbation of the impacts phase.

Substituting value $\tau = 2\pi n + \delta_\nu$ into this system we find the coordinates $x_{1e}^{(\nu)}$ and $x_{2e}^{(\nu)}$ of both parts of the system at the end of the ν-th interval, i.e., at the moment of the $(\nu+1)$th collisions. Then, substituting $\tau = 2\pi n$ in (9.23), we find coordinates x_{1e} and x_{2e} in the end of the interval of unperturbed motion. Differences /337

$$\Delta x_{1e}^{(\nu)} = x_{1e}^{(\nu)} - x_{1e}, \qquad \Delta x_{2e}^{(\nu)} = x_{2e}^{(\nu)} - x_{2e}$$

represent perturbations of coordinates accumulated by the system during the entire time of the perturbed motion. They prove equal to

$$\left.\begin{aligned} \Delta x_{1e}^{(\nu)} &= (\cos 2\pi n\zeta_1)\,\varepsilon_{1\nu}+(\sin 2\pi n\zeta_1)\,\varepsilon_{2\nu}+ \\ &+\Big(-\zeta_1 C_1\sin 2\pi n\zeta_1+\zeta_1 C_2\cos 2\pi n\zeta_1- \\ &\quad -\frac{\zeta_1^2}{\zeta_1^2-1}\sin\varphi\Big)\Delta_\nu- \\ &\quad -(-\zeta_1 C_1\sin 2\pi n\zeta_1+\zeta_1 C_2\cos 2\pi n\zeta_1)\,\Delta_{\nu-1}, \\ \Delta x_{2e}^{(\nu)} &= (\cos 2\pi n\zeta_2)\,\varepsilon_{3\nu}+(\sin 2\pi n\zeta_2)\,\varepsilon_{4\nu}+ \\ &+(-\zeta_2 C_3\sin 2\pi n\zeta_2+\zeta_2 C_4\cos 2\pi n\zeta_2)(\Delta_\nu-\Delta_{\nu-1}). \end{aligned}\right\} \quad (9.31)$$

As it follows from the analysis of the unperturbed motion (see preceding paragraph), the coefficients at Δ_ν and $\Delta_{\nu-1}$ in the first one of the equations in (9.31) are respectively equal to

$$-\zeta_1 C_1\sin 2\pi n\zeta_1+\zeta_1 C_2\cos 2\pi n\zeta_1-\frac{\zeta_1^2}{\zeta_1^2-1}\sin\varphi = v_1 = -\frac{2+\mu-\mu R}{\mu(1+R)}\lambda,$$

$$-\zeta_1 C_1\sin 2\pi n\zeta_1+\zeta_1 C_2\cos 2\pi n\zeta_1 = v_1+\frac{\zeta_1^2}{\zeta_1^2-1}\sin\varphi = -\frac{1}{\mu}\lambda.$$

In exactly the same way, from the relationships obtained in the preceding paragraph, it follows that the coefficient at the difference $(\Delta_\nu - \Delta_{\nu-1})$ in

the second equation of (9.31) is equal to

$$-\zeta_2 C_3 \sin 2\pi n\zeta_2 + \zeta_2 C_4 \cos 2\pi n\zeta_2 = \lambda.$$

Substituting this value in (9.31) we get /338

$$\left.\begin{aligned}\Delta x_{1e}^{(\nu)} &= (\cos 2\pi n\zeta_1)\,\varepsilon_{1\nu} + (\sin 2\pi n\zeta_1)\,\varepsilon_{2\nu} - \\ &\quad - \frac{2+\mu-\mu R}{\mu(1+R)}\lambda\Delta_\nu - \frac{1}{\mu}\lambda\Delta_{\nu-1}, \\ \Delta x_{2e}^{(\nu)} &= (\cos 2\pi n\zeta_2)\,\varepsilon_{3\nu} + (\sin 2\pi n\zeta_2)\,\varepsilon_{4\nu} + \\ &\quad + \lambda\Delta_\nu - \lambda\Delta_{\nu-1}.\end{aligned}\right\} \qquad (9.32)$$

Now we shall set up the expressions for the velocities of the perturbed motion. Differentiating (9.30) we get

$$\dot{x}_1^{(\nu)} = -\zeta_1(C_1+\varepsilon_{1\nu})\sin\zeta_1\tau + \zeta_1(C_2+\varepsilon_{2\nu})\cos\zeta_1\tau - \frac{\zeta_1^2}{\zeta_1^2-1}\sin(\tau+\varphi+\Delta_{\nu-1}),$$

$$\dot{x}_2^{(\nu)} = -\zeta_2(C_3+\varepsilon_{3\nu})\sin\zeta_2\tau + \zeta_2(C_4+\varepsilon_{4\nu})\cos\zeta_2\tau.$$

Substituting $\tau = 2\pi n + \delta_\nu$ here, we shall find the values of velocities $\dot{x}_{1e}^{(\nu)}$ and $\dot{x}_{2e}^{(\nu)}$ of both sides of the system at the end of the ν-th interval examined of the perturbed motion. Substituting $\tau = 2\pi n$ in (9.24) we find $\dot{x}_{1e}$ and $\dot{x}_{2e}$ at the end of the period of the unperturbed motion.

The differences

$$\Delta\dot{x}_{1e}^{(\nu)} = \dot{x}_{1e}^{(\nu)} - \dot{x}_{1e},$$

$$\Delta\dot{x}_{2e}^{(\nu)} = \dot{x}_{2e}^{(\nu)} - \dot{x}_{2e}$$

represent the accumulated velocity perturbations. They are equal to

$$\left.\begin{aligned}\Delta\dot{x}_{1e}^{(\nu)} &= (\zeta_1\sin 2\pi n\zeta_1)\,\varepsilon_{1\nu} + (\zeta_1\cos 2\pi n\zeta_1)\,\varepsilon_{2\nu} - \\ &\quad - \left(C_1\cos 2\pi n\zeta_1 + C_2\sin 2\pi n\zeta_1 + \frac{\cos\varphi}{\zeta_1^2-1}\right)\zeta_1^2\Delta_\nu + \\ &\quad + (C_1\cos 2\pi n\zeta_1 + C_2\sin 2\pi n\zeta_1)\,\zeta_1^2\Delta_{\nu-1}, \\ \Delta\dot{x}_{2e}^{(\nu)} &= (\zeta_2\sin 2\pi n\zeta_2)\,\varepsilon_{3\nu} + (\zeta_2\cos 2\pi n\zeta_2)\,\varepsilon_{4\nu} - \\ &\quad - (C_3\cos 2\pi n\zeta_2 + C_4\sin 2\pi n\zeta_2)\,\zeta_2^2(\Delta_\nu - \Delta_{\nu-1}).\end{aligned}\right\} \qquad (9.33)$$

Using the results obtained in the preceding paragraph we can easily show that the coefficients with Δ_ν and $\Delta_{\nu-1}$ in the first equation, as well as the /339 coefficient with the difference $(\Delta_\nu - \Delta_{\nu-1})$ in the second equation of system (9.33) are respectively equal to

$$\left.\begin{aligned}
&\left(C_1\cos 2\pi n\zeta_1 + C_2\sin 2\pi n\zeta_1 + \frac{\cos\varphi}{\zeta_1^2-1}\right)\zeta_1^2 = \\
&\qquad = \left(\frac{\zeta_1\cot\pi n\zeta_1}{\mu} - A\right)\lambda + L, \\
&(C_1\cos 2\pi n\zeta_1 + C_2\sin 2\pi n\zeta_1)\,\zeta_1^2 = \frac{\zeta_1\cot\pi n\zeta_1}{\mu}\,\lambda, \\
&(C_3\cos 2\pi n\zeta_2 - C_4\sin 2\pi n\zeta_2)\,\zeta_2^2 = (\zeta_2\cot\pi n\zeta_2)\,\lambda.
\end{aligned}\right\} \qquad (9.34)$$

Now we write the laws of perturbed motion in the following, $(\nu + 1)$th interval:

$$\left.\begin{aligned}
x_1^{(\nu+1)} &= (C_1 + \varepsilon_{1(\nu+1)})\cos\zeta_1\tau + \\
&\quad + (C_2 + \varepsilon_{2(\nu+1)})\sin\zeta_1\tau - \\
&\quad - \frac{L}{1+\varkappa} + \frac{\zeta_1^2}{\zeta_1^2-1}\cos(\tau + \varphi + \Delta_\nu), \\
x_2^{(\nu+1)} &= (C_3 + \varepsilon_{3(\nu+1)})\cos\zeta_2\tau + \\
&\quad + (C_4 + \varepsilon_{4(\nu+1)})\sin\zeta_2\tau + \frac{\varkappa L}{1+\varkappa}\,.
\end{aligned}\right\} \qquad (9.35)$$

Substituting value $\tau = 0$ here, we get the values of coordinates $x_{1b}^{(\nu+1)}$ and $x_{2b}^{(\nu+1)}$ at the beginning of the $(\nu + 1)$th interval of the perturbed motion, and then we shall find the perturbations of the coordinates

$$\left.\begin{aligned}
\Delta x_{1b}^{(\nu+1)} &= x_b^{(\nu+1)} - x_{1b} = \varepsilon_{1(\nu+1)} - \lambda\Delta_\nu, \\
\Delta x_{2b}^{(\nu+1)} &= x_{2b}^{(\nu+1)} - x_{2b} = \varepsilon_{3(\nu+1)}.
\end{aligned}\right\} \qquad (9.36)$$

Differentiating (9.35), substituting $\tau = 0$ in the expressions obtained, and making use of the results of the analysis of the unperturbed motion, we find the perturbations of velocities at the beginning of the $(\nu + 1)$th perturbed motion interval

$$\left.\begin{aligned}
\Delta\dot{x}_{1b}^{(\nu+1)} &= \zeta_1\varepsilon_{2(\nu+1)} - (L - A\lambda)\,\Delta_\nu, \\
\Delta\dot{x}_{2b}^{(\nu+1)} &= \zeta_2\varepsilon_{4(\nu+1)}.
\end{aligned}\right\} \qquad (9.37)$$

Now it remains for us to "fit" the ν-th and $(\nu + 1)$th intervals of perturbed motion one to another at the moment of collision: /340

$$\left.\begin{aligned}
&\Delta x_{1e}^{(\nu)} = \Delta x_{1b}^{(\nu+1)};\ \Delta x_{2e}^{(\nu)} = \Delta x_{2b}^{(\nu+1)};\ \Delta x_{1e}^{(\nu+1)} = \Delta x_{2b}^{(\nu+1)}, \\
&\Delta\dot{x}_{1b}^{(\nu+1)} = \frac{\mu - R}{1+\mu}\,\Delta\dot{x}_{1e}^{(\nu)} + \frac{1+R}{1+\mu}\,\Delta\dot{x}_{2e}^{(\nu)}, \\
&\Delta\dot{x}_{2b}^{(\nu+1)} = \frac{\mu(1+R)}{1+\mu}\,\Delta\dot{x}_{1e}^{(\nu)} + \frac{1-\mu R}{1+\mu}\,\Delta\dot{x}_{2e}^{(\nu)}.
\end{aligned}\right\} \qquad (9.38)$$

Substituting the magnitudes of the values of perturbation from (9.32), (9.33), (9.36) and (9.37) we shall obtain the following system of equations connecting values $\varepsilon_{i\nu}$, $\varepsilon_{i(\nu+1)}$, Δ_ν (i = 1, 2, 3, 4):

$$\left.\begin{aligned}
&\varepsilon_{1(\nu+1)} - (\cos 2\pi n\zeta_1)\varepsilon_{1\nu} + (\sin 2\pi n\zeta_1)\,\varepsilon_{2\nu} + \\
&\qquad + \frac{\lambda}{\mu}\Delta_\nu - \frac{\lambda}{\mu}\Delta_{\nu-1} = 0, \\
&\varepsilon_{3(\nu+1)} - (\cos 2\pi n\zeta_2)\,\varepsilon_{3\nu} - (\sin 2\pi n\zeta_2)\,\varepsilon_{4\nu} - \\
&\qquad - \lambda\Delta_\nu + \lambda\Delta_{\nu-1} = 0; \\
&\varepsilon_{1(\nu+1)} - \varepsilon_{3(\nu+1)} - b\lambda\Delta_\nu = 0, \\
&\mu\zeta_1(1+R)(\sin 2\pi n\zeta_1)\,\varepsilon_{1\nu} - \\
&\qquad - \mu\zeta_1(1+R)(\cos 2\pi n\zeta_1)\,\varepsilon_{2\nu} + \\
&\qquad + \zeta_2(1-\mu R)(\sin 2\pi n\zeta_2)\,\varepsilon_{3\nu} - \\
&- \zeta_2(1-\mu R)(\cos 2\pi n\zeta_2)\,\varepsilon_{4\nu} + \zeta_2(1+\mu)\,\varepsilon_{4(\nu+1)} + \\
&\quad + \{\mu(1+R)L - [N_1 + \mu(1+R)]A\}\lambda\Delta_\nu + \\
&\qquad + N_1\lambda\Delta_{\nu-1} = 0, \\
&\zeta_1(\mu-R)(\sin 2\pi n\zeta_1)\,\varepsilon_{1\nu} + \zeta_1(1+\mu)\,\varepsilon_{2(\nu+1)} - \\
&\qquad - \zeta_1(\mu-R)(\cos 2\pi n\zeta_1)\,\varepsilon_{2\nu} + \\
&\qquad + \zeta_2(1+R)(\sin 2\pi n\zeta_2)\,\varepsilon_{3\nu} - \\
&\qquad - \zeta_2(1+R)(\cos 2\pi n\zeta_2)\,\varepsilon_{4\nu} - \\
&- \{L(1+R) - [(1+R)A - N_2]\}\Delta_\nu + \lambda N_2\Delta_{\nu-1} = 0
\end{aligned}\right\} \qquad (9.39)$$

In the last two equations /341

$$N_1 = -(1-R)\zeta_1\cot\pi n\zeta_1 + (1-\mu R)\zeta_2\cot\pi n\zeta_2,$$
$$N_2 = \left(\frac{R}{\mu} - 1\right)\zeta_1\cot\pi n\zeta_1 + (1+R)\zeta_2\cot\pi n\zeta_2.$$

System (9.39) is a homogeneous system of equations in finite differences with constant coefficients. According to the method mentioned in Chapter 2, the solution of this system should be sought in the form of $\varepsilon_{i\nu} = b_i\beta^\nu$, $\Delta_\nu = b_5\beta^\nu$ (i = 1, 2, 3, 4).

Substituting these solutions into system (9.39) we obtain the characteristic equation in the form of a determinant of the fifth order. We shall omit the transformations connected with the reduction of the order of this determinant. We shall mention only that as their result we can lower its order to the third, and reduce it to the following form:

$$\begin{vmatrix} c_{11}\beta + a_{11} & a_{12} & c_{13}\beta + a_{13} \\ c_{21}\beta + a_{21} & c_{22}\beta + a_{22} & c_{23}\beta + a_{23} \\ d_{31}\beta^2 + c_{31}\beta + a_{13} & c_{32}\beta & d_{33}\beta^2 + c_{33}\beta + a_{33} \end{vmatrix} = 0. \qquad (9.40)$$

In this determinant

$$a_{11} = -\cos 2\pi n\zeta_2,\ a_{21} = \zeta_2(1-\mu R)\sin 2\pi n\zeta_2 + \frac{\mu(1+R)\zeta_1}{\sin 2\pi n\zeta_1},$$

$$a_{31} = -\frac{R(1+\mu)\zeta_1}{\sin 2\pi n\zeta_1},\ a_{12} = -\sin 2\pi n\zeta_2,$$

$$a_{22} = (\mu R - 1)\zeta_2 \cos 2\pi n\zeta_2,\ a_{13} = \lambda,$$

$$a_{23} = \left[(1-\mu R)\zeta_2 \cot \pi n\zeta_2 + \frac{\mu(1-R)-2R}{\sin 2\pi n\zeta_1}\right]\lambda,$$

$$a_{33} = \frac{R}{\mu}\frac{1+\mu}{1+R}\frac{2R-\mu(1-R)}{\sin 2\pi n\zeta_1}\zeta_1\lambda,$$

$$c_{11} = 1,\ c_{21} = -\mu(1+R)\zeta_1 \cot 2\pi n\zeta_1,$$

$$c_{31} = (R+2R\mu-1)\zeta_1 \cot 2\pi n\zeta_1,$$

$$c_{22} = (1+\mu)\zeta_2,\ c_{32} = -(1+R)\zeta_2,\ c_{13} = -\lambda,$$

$$c_{23} = [(1+R)\zeta_1 \cot \pi n\zeta_1 - (2+\mu-\mu R)\zeta_1 \cot 2\pi n\zeta_1 -$$
$$-(1-\mu R)\zeta_2 \cot \pi n\zeta_2]\lambda - \mu(1+R)(A\lambda - L),$$

and /342

$$c_{33} = (1+R)(A\lambda - L) - \frac{(1+R)\zeta_1\lambda}{\mu \sin 2\pi n\zeta_1} - (1-R-2R\mu)b\zeta_1\lambda \cot 2\pi n\zeta_1,$$

$$d_{31} = \frac{(1-\mu R)\zeta_1}{\sin 2\pi n\zeta_1},\ d_{33} = \left(\frac{1}{\mu}+b\right)\frac{(1-\mu R)\zeta_1\lambda}{\sin 2\pi n\zeta_1}.$$

The determinant (9.40) represents the characteristic equation of the system examined. Expanded, it assumes the following form:

$$a_0\beta^4 + a_1\beta^3 + a_2\beta^2 + a_3\beta + a_4 = 0. \tag{9.41}$$

Coefficients a_i are expressed with parameters of the system and actuation in the following way:

$$\left.\begin{aligned} a_0 &= \frac{2(1+\mu)^2}{(1+R)\mu}\frac{\lambda}{\sin 2\pi n\zeta_1}, \\ a_1 &= -\left(\frac{P_1}{\sin 2\pi n\zeta_1} + P_2\right)\lambda - LP_3, \\ a_2 &= -\lambda P_4 + LP_5,\ a_3 = -\lambda P_2 - LP_3,\ a_4 = R^2 a_0. \end{aligned}\right\} \tag{9.42}$$

Here

$$P_1 = 4(1+\mu)(1-R)\left(\cos 2\pi n\zeta_1 + \frac{1}{\mu}\cos 2\pi n\zeta_2\right),$$

$$P_2 = -\left(\frac{1+R}{\zeta_1} + \mu\frac{1+R}{\zeta_2}\frac{\sin 2\pi n\zeta_2}{\sin 2\pi n\zeta_1}\right)A + \frac{(1+R)(1+\mu^2)}{\mu \sin 2\pi n\zeta_1} +$$
$$+\frac{[2R-\mu(1-R)]^2}{\mu(1+R)}\frac{\cos 2\pi n\zeta_2}{\sin 2\pi n\zeta_1} + \frac{(R-1+2\mu R)^2}{\mu(1+R)}\cot 2\pi n\zeta_1 +$$
$$+(1+R)\sqrt{\frac{\varkappa}{\mu}}\frac{\sin 2\pi n\zeta_1}{2\sin^2 \pi n\zeta_1} + (1+R)\sqrt{\frac{\mu}{\varkappa}}\cot \pi n\zeta_2,$$

$$P_3 = \frac{1+R}{\zeta_1} + \frac{\mu(1+R)}{\zeta_2} \cdot \frac{\sin 2\pi n\zeta_2}{\sin 2\pi n\zeta_1},$$

$$P_4 = -2 - \frac{(1-\mu R)^2+(R-\mu)^2}{\mu(1+R)\sin 2\pi n\zeta_1} + 2\left[\frac{1+R}{\zeta_1}\cos 2\pi n\zeta_2 + \right.$$

$$\left. + \frac{\mu(1+R)}{\zeta_2}\sin 2\pi n\zeta_2 \cot 2\pi n\zeta_1\right] A - 2(1+R)\sqrt{\frac{\mu}{\varkappa}}\cot \pi n\zeta_2 -$$

$$- \frac{(1-R)^2(1+\mu)^2}{(1+R)\mu}\cos 2\pi n\zeta_2 \cot 2\pi n\zeta_1 - 2\mu(1+R)\cot 2\pi n\zeta_1 - \qquad /343$$

$$- \frac{2(1+R)}{\mu}\frac{\cos 2\pi n\zeta_2}{\sin 2\pi n\zeta_1} - (1+R)\sqrt{\frac{\varkappa}{\mu}}\frac{\sin 2\pi n\zeta_2}{\sin^2 \pi n\zeta_1},$$

$$P_5 = \frac{2(1+R)}{\zeta_1}\cos 2\pi n\zeta_2 + \frac{2\mu(1+R)}{\zeta_2}\sin 2\pi n\zeta_2 \cot 2\pi n\zeta_1.$$

Values b and A encountered in (9.40) and (9.42) are the same as in ratios (8.43) and (9.27).

§ 9.5. The Stability Criterion.

The problem of investigation of the stability of the problem of the periodic conditions of motion found is reduced to the determination of the parameter regions for which all the roots of equation (9.41) are less than one with respect to the modulus. To solve this problem we use Schur's inequality. In our designations these inequalities will be written as follows:

$$W_0 = R^2 < 1, \tag{9.43}$$

$$W_1 = \left|\frac{S_3}{S_6}\right| < 1, \tag{9.44}$$

$$W_2 = \left|\frac{S_1}{S_2}\right| < 1, \tag{9.45}$$

$$W_3 = \left|\begin{matrix} S_3S_4 + S_5S_6 \\ S_1 \end{matrix}\right| \left|\begin{matrix} \\ S_2 \end{matrix}\right| < 1, \tag{9.46}$$

where

$$S_1 = (a_1R^2 - a_3)(a_1 - a_3R^2) + (1+R^2)(1-R^2)^2 a_0a_2,$$
$$S_2 = a_0(1-R^2)^2 - (a_3 - a_1R^2)^2, \quad S_3 = a_3 - a_1R^2,$$
$$S_4 = a_2(R^2-1), \quad S_5 = a_1 - a_3R^2, \quad S_6 = a_0(1-R^2).$$

The regions of stable solutions encompass the values of parameters satisfying simultaneously all the four inequalities.

The first inequality is fulfilled for the entire range of variations $0 \leq R < 1$, with the exception of case R = 1. The regions of the parameters satisfying the three remaining inequalities, will be determined in the following manner.

Instead of inequalities (9.44), (9.45) and (9.46) we shall solve the exact equalities and in this way find the relationships, corresponding to the limits separating the regions of stable and unstable solutions. Then, with direct evaluation we shall establish in which direction from one or the other boundary found are located the regions of the parameters, satisfying the corresponding inequality. /344

Let us examine the equation

$$W_1 = \left| \frac{a_3 - R^2 a_1}{(1 - R^4)\, a_0} \right| = 1.$$

corresponding to the inequality (9.44). From it proceed the following two relationships:

$$\frac{a_3 - R^2 a_1}{(1 - R^4)\, a_0} = 1; \tag{9.47}$$

$$\frac{a_3 - R^2 a_1}{(1 - R^4)\, a_0} = -1. \tag{9.48}$$

We substitute into equation (9.47) the value of coefficients a_0, a_1 and a_3 according to (9.42). After the transformation we shall obtain the following relationships:

$$\lambda = \frac{D_2}{D_1} L. \tag{9.49}$$

Here

$$D_1 = \frac{(1 + R) R^2 \mu P_1 - 2(1 - R^2)(1 + \mu)^2}{(1 - R)\mu \sin 2\pi n \zeta_1} + (R^2 - 1) P_2,$$

$$D_2 = (1 - R^2) P_3.$$

We substitute in (9.49) instead of λ, its expression according to (9.26), and solve the relationship found with respect to L:

$$L = bB \sqrt{\left[(A^2 + b^2) \frac{D_2}{D_1} - A \right]^2 + b^2}. \tag{9.50}$$

This formula, as well as the analogous formulas, which we shall obtain from the other inequalities, connects five independent parameters: μ, κ, R, L, ζ_1 (or ζ_2). In such instances the solution of the problem of determination of the regions of stability of the periodic conditions of motions with exhaustive completeness, is connected with extremely great difficulties. Therefore we shall act here, as we did in the preceding chapter in the analysis of the stability of periodic conditions of motion of the nonlinear element with a clearance. We fix the parameters μ, κ, and R. In this case formula (9.50) /345

may be examined as the equation $L = L(\zeta_1)$ of the boundary curve, separating the regions, within the bounds of which $W_1 > 1$ and $W_1 < 1$. In order to find on which side of the boundary curve the $W_1 < 1$ region is located, into the initial equation, determining W_1, instead of the boundary values L_{bd} we substitute values L, which are close to L_{bd}. If,when $L > L_{bd}$ we find that $W_1 < 1$, and when $L < L_{bd}$ we have $W_1 > 1$, then the region sought is located above the boundary curve. If with $L > L_{bd}$ $W_1 > 1$ and with $L < L_{bd}$ $W_1 < 1$, then it is located below the boundary curve. We should note that functions of W_i depend in a complex manner on L, and in order to solve unerringly the ranges of variations of L, satisfying each inequality $|W_i| < 1$, a thorough analysis of these functions is required, in particular in order to determine the points of rupture.

According to (9.26) λ is an ambiguous function. Which of the two values of λ satisfies the equality $W_1 = 1$, is determined by their direct substitution in (9.49).

We have examined the method of analysis of the extreme equality (9.47) because it may be used in the examination of all the other stability criteria. Thus, in particular, for the maximum equation (9.48), acting according to the method indicated, we get

$$\lambda = \frac{D_2}{D_3} L, \tag{9.51}$$

where

$$D_3 = -\frac{(1+R) R^2 \mu P_1 + 2(1-R^4)(1+\mu)^2}{(1+R)\mu \sin 2\pi n \zeta_1} + (R^2-1) P_2.$$

Substituting the value λ here and solving the obtained equation with respect to L, we find

$$L = \pm bB \sqrt{\frac{A^2+b^2}{\left[(A^2+b^2)\frac{D_2}{D_3} - A\right]^2 + b^2}}. \tag{9.52}$$

Curve $L = L(\zeta_1)$ plotted according to this formula is the boundary curve and corresponds to the values $W_1 = -1$. The question of to which side of this /346 curve is the $W_1 > 1$ region located is solved in precisely the same way as it was solved earlier, applicably to formula (9.50).

Now we shall start the search for the regions of parameters satisfying

inequality (9.45). For this purpose we shall first examine the exact equalities

$$W_2 = 1, \quad W_2 = -1. \tag{9.53}$$

The first of those equalities results in the following relationship:

$$\lambda = \frac{D_5}{D_4} L, \tag{9.54}$$

$$D_4 = \left[\frac{R^2 P_1}{\sin 2\pi n \zeta_1} + (R^2 - 1) P_2 \right] P_1 - \left[\frac{1 + R^2}{1 + R} \frac{2(1+\mu)^2}{\mu \sin 2\pi n \zeta_1} + P_4 \right] \frac{2(1-R^2)^2 (1+\mu)^2}{(1+R)\mu},$$

$$D_5 = \frac{(1-R^2) P_1 P_3}{\sin 2\pi n \zeta_1} - \frac{(1-R^2)^2}{1+R} \frac{2(1+\mu)^2}{\mu \sin 2\pi n \zeta_1} P_5.$$

For the boundary curve $L(\zeta_1)$ we get the equation analogous to (9.50) and (9.52):

$$L = \pm bB \sqrt{\frac{A^2 + b^2}{\left[(A^2 + b^2) \frac{D_5}{D_4} - A \right]^2 + b^2}}. \tag{9.55}$$

Applying the above-mentioned method we determine the region, within the bounds of which $W_2 < +1$. For the first equation in (9.53) after substitution of the values of the coefficients we get

$$(R^2 a_1 - a_3)(a_1 + a_3) + (1 - R^4) a_0 [a_2 + a_0 (1 + R^2)] = 0. \tag{9.56}$$

The subsequent transformations with the substitution of values a_i result in the following equation:

$$b_0 \lambda^2 + b_1 L \lambda + b_2 L^2 = 0, \tag{9.57}$$

$$b_0 = \left[(1 - R^2) P_2 - \frac{R^2 P_1}{\sin 2\pi n \zeta_1} \right] \left[2P_2 + \frac{P_1}{\sin 2\pi n \zeta_1} \right] +$$

$$+ \left[P_4 - \frac{2(1+\mu)^2 (1+R^2)}{\mu (1+R)} \cdot \frac{1}{\sin 2\pi n \zeta_1} \right] \frac{2(1+\mu)^2 (1-R^4)}{\mu (1+R) \sin 2\pi n \zeta_1},$$

$$b_1 = (1 - R^2) \left(2P_2 + \frac{P_1}{\sin 2\pi n \zeta_1} \right) P_3 + \qquad \text{/347}$$

$$+ \left[(1 - R^2) P_2 - \frac{R^2 P_1}{\sin 2\pi n \zeta_1} \right] 2P_3 - \frac{2(1+\mu)^2 (1-R^4) P_5}{\mu (1+R) \sin 2\pi n \zeta_1},$$

$$b_2 = 2(1 - R^2) P_3^2.$$

Let us solve this equation with respect to λ

$$\lambda = D_{6,7} L,$$

$$D_{6,7} = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_0 b_2}}{2b_0}. \tag{9.58}$$

The root of D_6 corresponds to the plus sign, D_7 to the minus sign. Substituting in (9.57) the expression for λ, found for the unperturbed motion, and solving the equation obtained for L, we find

$$L = \pm bB \sqrt{\frac{A^2 + b^2}{[(A^2 + b^2) D_{6,7} - A]^2 + b^2}}. \tag{9.59}$$

Each of the curves $L(\zeta_1)$ calculated according to this formula, connects the points for which $W_2 = -1$. The region of values, for which $W_2 > -1$, is determined, as in all the preceding instances, by direct verification.

Now, it remains to determine the boundary curves $L(\zeta_1)$, for which $W_3 = +1$ and $W_3 = -1$.

According to (9.46) for the first of these equations we shall have

$$W_3 = \frac{S_3 S_4 + S_5 S_6}{S_1 + S_2} = +1.$$

Substituting values S_i (i = 1, 2, ..., 6) here, and performing the required computations, we get

$$\begin{aligned} &(1 + R^2) a_0 (a_1 - R^2 a_3) - a_2 (a_3 - R^2 a_1) = \\ &= (R^2 a_1 - a_3)(a_1 + a_3) + (1 - R^4) a_0 [a_2 + a_0 (1 + R^2)]. \end{aligned} \tag{9.60}$$

The right part of this equation coincides exactly with the left part of (9.56). Therefore, (9.60) may be rewritten in the following way:

$$(1 + R^2) a_0 (a_1 - R^2 a_3) - a_2 (a_3 - R^2 a_1) = b_0 \lambda^2 + b_1 L \lambda - b_2 L^2.$$

Substituting the a_i (i = 1, 2, 3) values here we get

$$b_3 \lambda^2 + b_4 L \lambda + b_5 = 0, \tag{9.61}$$

where /348

$$b_3 = -\frac{2(1+\mu)^2 (1+R^2)}{\mu (1+R) \sin 2\pi n \zeta_1} \left[(1 - R^2) P_2 + \frac{P_1}{\sin 2\pi n \zeta_1} \right] - \left[(1 - R^2) P_2 - \frac{R^2 P_1}{\sin 2\pi n \zeta_1} \right] P_4 - b_0,$$

$$b_4 = -(1-R^4)\frac{2(1+\mu)^2 P_3}{\mu(1+R)\sin 2\pi n\zeta_1} + (1-R^2)P_2P_5 - \frac{R^2P_1P_5}{\sin 2\pi n\zeta_1} - (1-R^2)P_3P_4 - b_1,$$

$$b_5 = (1-R^2)P_3P_5 - b_2.$$

We solve equation (9.61) with respect to λ:

$$\lambda = D_{8,9}L, \tag{9.62}$$

where

$$D_{8,9} = \frac{-b_4 \pm \sqrt{b_4^2 - 4b_3b_5}}{2b_3}.$$

Here, coefficient D_8 corresponds to the plus sign before the radical, D_9 to the minus sign. Acting in accordance with the conventional methods, we get the equations of the boundary curves

$$L = \pm bB\sqrt{\frac{A^2+b^2}{(A^2+b^2)D_{8,9}-A]^2+b^2}}, \tag{9.63}$$

connecting the points for which $W_3 = +1$. The boundary curves for $W_3 = -1$ are determined similarly. Without repeating the computations, we reduce the corresponding equation, analogous to equation (9.61):

$$b_6\lambda^2 + b_7L\lambda + b_8 = 0. \tag{9.64}$$

Here, coefficient b_6, b_7, b_8 differ from coefficients b_3, b_4, b_5 included in (9.61) only by the fact that before components b_0, b_1, b_2 instead of the minus sign stands the plus sign.

Solving equation (9.64) with respect to λ, we find

$$\lambda = D_{10,11}L,$$

$$D_{10,11} = \frac{-b_7 \pm \sqrt{b_7^2 - 4b_6b_8}}{2b_6}. \tag{9.65}$$

The equation of boundary curves for $W_3 = 1$ has the following appearance

$$L = \pm bB\sqrt{\frac{A^2+b^2}{(A^2+b^2)D_{10,11}-A]^2+b^2}}. \tag{9.66}$$

In this way, making use of Schur's criterion we shall find a number of structurally similar relationships (9.50), (9.52), (9.59), (9.63), (9.66), which enable us to separate on plane (L, ζ_1) the regions within the bounds of /349

which the periodical conditions of the motion of a two-mass vibro-impact system prove stable. We remind the reader once again that values of L and ζ_1 corresponding to these regions, satisfy simultaneously three conditions of stability (9.44), (9.45), and (9.46).

In Figure 9.8 as an example we show a section of the stability chart, plotted for fixed values: $\mu = 2$, $\kappa = 1$, $R = 0.5$, $n = 1$, in the $0.6 \leq \zeta_1 \leq 0.8$ range.

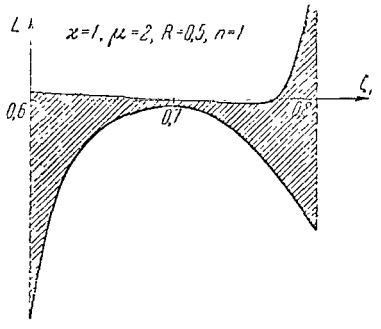

Figure 9.8.

In a similar way we may plot the regions of stability for the periodic conditions $n = 2$, $n = 3$, etc.

In the period of values of ζ_1, represented in Figure 9.8, those conditions of motion proved stable, which correspond to parameter λ, calculated with formula (9.26), with the plus sign before the radical. However, the possibility is not excluded, that with certain values of ζ_1 and L, such conditions will also prove stable, which correspond to the values of λ with the minus sign before the root in formula (9.26). Thus, for example, when $\zeta_1 = 0.8$, in the range of

$$1.81 < L < 3.00$$

both conditions prove stable. Consequently, the analysis of stability does not always solve unambiguously the problem of the system's movement.

§ 9.6. Collisions in Systems Operating Under Vibration Conditions.

In this paragraph we shall return to the range of problems which was already examined in the fifth and sixth chapters of this book. It is connected with the investigation of the dynamics and stability of systems /350
with elastic couplings, operating under the conditions of steady vibration. An appreciable place there was occupied by the problems of determination of the dynamic errors, originating as the result of imbalance of the mobile components of the system and friction in its kinematic pairs. Along with the imbalance and friction a substantial source of dynamic errors may be the collisions in the kinematic pairs, which are connected by means of elastic couplings.

Thus, for example, in Figure 9.9 we give a conventional diagram of a measuring instrument, in which the transmission from the elastic sensor element to pointer 5 is performed by means of lever 2, cogged sector 3, and a small cog wheel (gear) 4, connected with the pointer. The free play accumulated in such a system may attain a substantial value and in order to

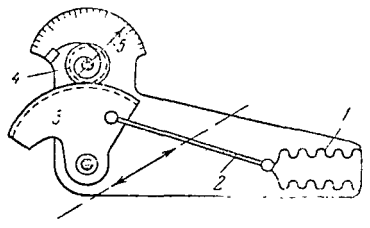

Figure 9.9.

exclude its effect on the accuracy of the instrument's readings, a small spiral spring is connected to the gear, the preliminary stress on which provides for a one-sided selection of free play in the system. However, upon the action of intense vibration on the instrument, in a number of cases this stress proves insufficient, and the oscillations of the mobile system of the instrument are accompanied by ruptures and collisions of the elements of its kinematic chain, and a characteristic rattle.

Numerous experiments have shown that such vibro-impact motions often result in an amazing, at first glance, effect. Under the action of harmonic vibrations, the pointer of the instrument oscillates not near the static equilibrium position, but at a certain displaced position, in other words, along with disalignment of the pointer, it is moved away. Our task, now, is to find out the causes of origination of this effect, and determine both components of the full dynamic error of the instrument. For this we shall resort to the examination of the dynamic model presented in Figure 9.10. As we can see, this model differs from the one given in Figure 9.1 only by the fact that the external actuation is applied to the body and not to one of the masses. Therefore, all that has been said in the first two paragraphs of this chapter with respect to the proper features of the model given in Figure 9.1 (conditions of origination of ruptures, frequency of free oscillations, etc) is completely applicable to the system in Figure 9.10. Therefore, we shall only analyze briefly its forced motions with periodic collisions, the frequency of which is equal to the frequency of the forcing force. This motion must satisfy the earlier-indicated periodicity conditions (9.25).

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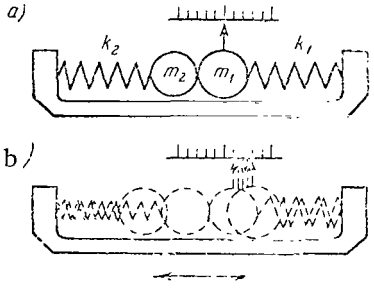

Figure 9.10.

The equations describing the motions of both parts of the system in the interval between collisions, during a harmonic vibration $\rho = \rho_0 \cos \omega t$ of the base, have the following appearance:

$$\ddot{x}_1^* + \omega_1^2 x_1^* = -\frac{f\omega_1^2}{1+\varkappa} + \rho_0\omega^2 \cos(\omega t + \varphi),$$

$$\ddot{x}_2^* + \omega_2^2 x_2^* = \frac{\varkappa l \omega_2^2}{1+\varkappa} + \rho_0\omega^2 \cos(\omega t + \varphi).$$

For the convenience of the further analysis we shall write their solution in the following form:

$$\left.\begin{aligned} x_1 &= -\frac{L}{1+\varkappa} + x_{11}\cos(\zeta_1\tau + \psi_1) + x_{12}\cos(\tau+\varphi), \\ x_2 &= \frac{\varkappa L}{1+\varkappa} + x_{21}\cos(\zeta_2\tau + \psi_2) + x_{22}\cos(\tau+\varphi). \end{aligned}\right\} \qquad (9.67)$$

We shall use the pulse theorem and periodicity conditions (9.25) and will act according to the same procedure as the one used in § 9.3, making all the computations on the assumption that n = 1.

Then we shall find: /352

$$\left.\begin{aligned} \psi_1 &= -\pi\zeta_1,\ x_{11} = \frac{x_{11}^*}{\varrho_0} = \frac{s_1\lambda}{2}\frac{1}{\zeta_1\sin\pi\zeta_1},\quad x_{12} = \frac{1}{\zeta_1^2-1}, \\ \psi_2 &= -\pi\zeta_2,\quad x_{21} = \frac{x_{21}^*}{\varrho_0} = -\mu\frac{s_1\lambda}{2}\frac{1}{\zeta_2\sin\pi\zeta_2}, \\ x_{22} &= \frac{1}{\zeta_2^2-1},\ \sin\varphi = -\frac{s_2\lambda}{2}(\zeta_1^2-1). \end{aligned}\right\} \qquad (9.68)$$

In (9.67) and (9.68) the following designations are adopted:

$$L = \frac{l}{\varrho_0};\quad \lambda = \frac{u_1}{\omega\varrho_0},\quad s_1 = \frac{1+R}{1+\mu}(1-\gamma),$$
$$s_2 = \frac{1+R}{1+\mu} + \frac{1-R+2\mu}{1+\mu}\gamma,\quad \gamma = \frac{v_1}{\lambda}\frac{(2-\mu R+\mu)E-(1+R)}{(1-R+2\mu)-\mu(1+R)E},$$
$$E = \frac{\zeta_2^2-1}{\zeta_1^2-1},\quad \zeta_1 = \frac{\omega_1}{\omega},\quad \zeta_2 = \frac{\omega_2}{\omega}.$$

Substituting the values found into (9.67) we obtain, with $\tau = 2\pi$, two equations, determining the dimensionless collision coordinate x_c and dimensionless velocity λ:

$$\left.\begin{aligned} x_c &= -\frac{L}{1+\varkappa} + \frac{s_1\lambda}{2\zeta_1}\cot\pi\zeta_1 + \frac{\cos\varphi}{\zeta_1^2-1}, \\ x_c &= \frac{\varkappa L}{1+\varkappa} - \frac{\mu s_1\lambda}{2\zeta_2}\cot\pi\zeta_2 + \frac{\cos\varphi}{\zeta_2^2-1}. \end{aligned}\right\} \qquad (9.69)$$

Having solved these equations jointly, we get

$$\lambda = \frac{\frac{2}{\mu}AL \pm b\sqrt{F^2(A^2+b^2) - \left(\frac{2}{\mu s_1}L\right)^2}}{A^2+b^2}. \qquad (9.70)$$

In this formula $F = \dfrac{2}{s_1(\zeta_1^2-1)}$ we designate with A and b the same values as in formula (9.26).

We remind the reader here of the substantial feature of the vibro-impact system examined, consisting of the fact that the motion of both ot its parts

is the result of superposition of free and forced oscillations. Parameter λ is the most convenient for the frequency analysis of such a system; it had been used already for these purposes.

If we attribute to the system the conservation properties (R = 1), then in $\gamma = -1/\mu$, $s_1 = 0$, $\sin\phi = 0$. According to the equalities (9.69) we get /353

$$\lambda = \frac{1}{A}\left[L + \frac{(1-E)(\varkappa E-\mu)}{2E(\mu-\varkappa)}\right]. \qquad (9.71)$$

It is not hard to see that from this formula we get the following formal result. Under conditions when the forcing frequency coincides with one of frequencies $\bar{\omega}_1$ of the free oscillations calculated when L = 0, the collision velocity increases to infinity, curve $\lambda = \lambda(\zeta)$ undergoes a break. For the non-conservative systems this characteristic has the usual resonance character, as shown in Figure 9.11. Here we present the characteristic $\lambda = \lambda(\zeta_2)$, plotted for the frequency range, encompassing the maximum possible $\bar{\omega}$ frequency. In Figure 9.12 we give a family of characteristics plotted for various magnitudes of the value of the initial tightness. The vertical lines drawn with a dash-dot line, indicate the values of $\bar{\omega}$ calculated for the same values of the parameters when R = 1. Here, λ achieves their maximum values with actuation frequencies, somewhat smaller than the corresponding frequencies $\bar{\omega}$.

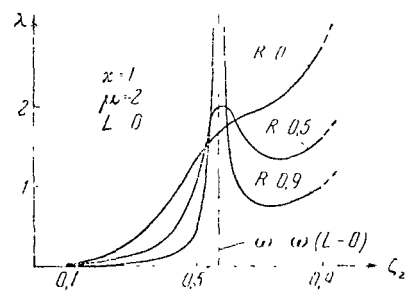

Figure 9.11.

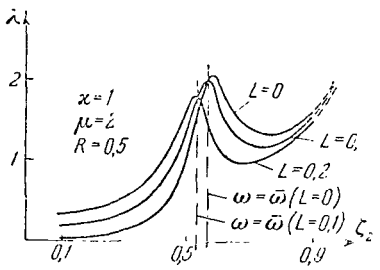

Figure 9.12.

Solutions (9.67) are current with any values of the actuation frequency, with the exception of instances when $\omega = \omega_1$, or $\omega = \omega_2$. With these ω values, formulas (9.70) and (9.71) result in an ambiguity. However, the conditions /354 indicated, in this case are not special in any sense. We can assure ourselves about this by opening the ambiguity either if we must completely determine the motion of a system, seeking the particular solutions in the form of $x_{12}\,\text{tsin}\,(\omega_1 t + \phi)$, or accordingly $x_{22}\,\text{tsin}\,(\omega_2 t + \phi)$. In either case we obtain the following result:

$$\lambda_{\zeta_1=1} = \frac{\pi\mu}{2}, \qquad \lambda_{\zeta_2=1} = \frac{\pi}{2}\frac{1-R+2\mu^2}{(1+R)\mu}.$$

With R = 1, both values of λ prove equivalent. For the extreme point of the characteristic, when $\zeta_1 \to 0$ ($\omega \to \infty$) we get $E \to \infty$, $\gamma \to 1$, $A \to \infty$. Turning to

formula (9.70) and opening the ambiguity we find

$$\lim_{\zeta_1 \to 0} \lambda = \frac{\pi\varkappa(1+\mu)(R-1)L}{(\varkappa-\mu)(1+\varkappa)(1+R)}.$$

Thus, with the increase of actuation frequency within the bounds of $\bar{\omega} < \omega < \infty$, collision velocity decreases continuously, divergence of both parts of the system in the intervals between collisions decreases, and their trajectories draw closer.

Plotting of frequency characteristics $\lambda = \lambda(\zeta_1)$ may be extended also in the direction of $\zeta_1 \to \infty$ ($\omega \to 0$), as it is shown in Figure 9.13. On this characteristic we mark three points, corresponding to values $\omega = \bar{\omega}$, $\omega = \omega_1$, and $\omega = \omega_2$. On the sector plotted the characteristic has two extreme values. We have already spoken about one maximum, as applied to the characteristic in Figures 9.11 and 9.12. It takes place in the vicinity of the first value of $\bar{\omega}$.

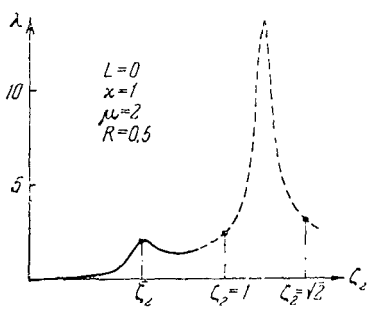

Figure 9.13.

In the analysis of the free oscillations it was shown that to the two-mass vibro-impact system formally corresponds in infinite number of roots of $\bar{\omega}_i$. The second extreme value of characteristic $\lambda = \lambda(\zeta_1)$ in Figure 9.13 pre- /355
cisely corresponds to the second root of the frequency equation (9.17). As we can see in the curve, with the selected values of the parameters, value λ grows sharply with the approach to the second resonance. If we extend the characteristics plotted up to the values $\omega \to 0$, evidently it will contain a number of splashes, corresponding to the successive values of $\bar{\omega}_i$. However, with the relatively low frequencies of actuation, the condition according to which in the region between the collisions x_1 should be greater than x_2, may be disrupted. Here, additional collisions will take place, disrupting the specified periodic character of the system's motion. One of such sets of conditions, formally satisfying the conditions of periodicity in (9.25) is given in Figure 9.14.

Having performed the analysis of the forced oscillations of the system examined, we may now turn to the determination of the values of its dynamic errors. In the fifth and sixth chapters, determining the dynamic errors originating as the result of the imbalance of the mobile system and friction in the kinematic pairs, we had mentioned the dynamic errors of the entire mechanism. In distinction from this, upon the examination of the two-mass vibro-impact system, we should mention the dynamic errors of separate parts of the system, the motion of which differs substantially.

Proceeding from this, under the forced shift of one or another part of the system we shall understand the difference

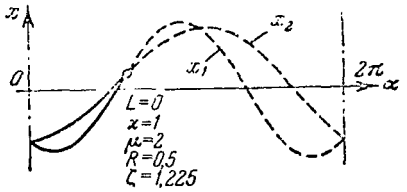

Figure 9.14.

$$\Delta_i = x_{\text{dyn } i} - x_0 \quad (i = 1, 2);$$

x_0 is the coordinate characterizing the position of static equilibrium of the system, and $x_{\text{dyn}} = \frac{1}{T}\int_0^T x_i(t)\,dt$ is the coordinate, characterizing the position of the dynamic equilibrium of one or another part of the system.

Bearing in mind that the beginning of the count was combined with the position of the static equilibrium, we get, in the dimensionless form /356

$$\delta_i = \frac{1}{2\pi}\int_0^{2\pi} x_i(\tau)\,d\tau; \quad \delta_i = \frac{\Lambda_i}{Q_0}.$$

At first we determine the forced shift of the entire system as a whole, conventionally characterizing its position by coordinate x_s of the center of inertia

$$x_s = \frac{\mu x_1 + x_2}{1+\mu}. \tag{9.72}$$

Using the laws of motion of (9.67) we find

$$\delta_s = \frac{\mu - \varkappa}{1+\mu}\left(\frac{s_1\lambda}{2\pi\zeta_1^2} - \frac{L}{1+\varkappa}\right). \tag{9.73}$$

The curves in Figure 9.15 are plotted according to this formula. As we can see the forced shift of the system is a linear function of parameter λ and accordingly, the frequency characteristic $\delta_s = \delta_s(\zeta_1)$ has about the same appearance as the $\lambda = \lambda(\zeta_1)$ characteristic. The characteristics in Figure 9.15 are plotted for the region of actuation frequencies, within the bounds of which frequency $\bar{\omega}$ of the free oscillations of the system is located. It would be possible to extend them toward the smaller actuation frequencies, however, this construction attains a sense only after the frequency range has been determined, within the bounds of which the specified periodic character of the system's motion is preserved. The effect of the system's being led away is

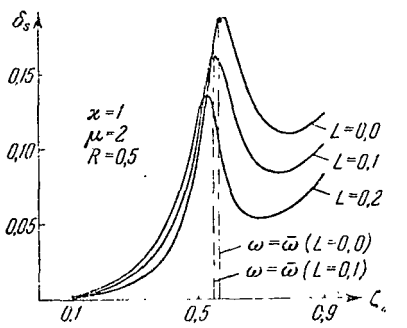

Figure 9.15.

caused by the impact interaction of the parts of the system, its substantial nonlinearity. The magnitude of the forced shift of the system grows with the increase of parameter λ and the coefficient of restitution upon impact.

When the frequencies of the free oscillations of individual parts of a system are equal, the $\kappa = \mu$ equality takes place. Here, value $\delta_s = 0$. This result agrees with that obtained at the beginning of this chapter through the conclusion on the impossibility of conditions of motions with ruptures and collisions when $\kappa = \mu$. /357

Having relationships (9.72) and (9.73) at our disposal it is not hard to determine the value of forced shifts of the individual parts of the system. They are found equal to

$$\delta_1 = \frac{1+\mu}{\mu-\kappa}\delta_s, \qquad \delta_2 = -\frac{1+\mu}{\mu-\kappa}\frac{\delta_s}{\kappa}.$$

Their ratio is equal to

$$\frac{\delta_2}{\delta_1} = -\kappa. \tag{9.74}$$

Thus, the forced shifts of both parts of the system are found to have opposite signs. Their relative value depends exclusively on the rigidities of the elastic couplings, and the greater the rigidity on the elastic coupling, imposed on the part of the mechanism examined, the less is its forced shift.

The determination of the value of the second component of the full dynamic error, the disalignment of individual parts of the system, under which we understand the double amplitude of their forced oscillations, in the presence of (9.68) relationships, determining the values of amplitudes of free and forced oscillations, presents no difficulties.

§ 9.7. Experiments with an Asymmetric Vibro-Impact System.

The method of the analysis of the stability of periodic conditions of motion of an asymmetrical two-mass vibro-impact system is fully applicable for the occasion examined here, when the external actuation is applied simultaneously to both parts of the system. We have already seen that such an analysis requires complex and cumbersome computations. It is hardly rational to repeat them, especially since no substantially new methods in comparison with the preceding case, will be required in the case examined. Therefore, we shall terminate the present chapter, as we did Chapter 8 in which symmetrical vibro-impact systems were examined, by presenting the results of experiments performed with a model, coinciding exactly with the one given in Figure 9.1.

In Figure 9.16 we present the photograph of this model, installed on the /358

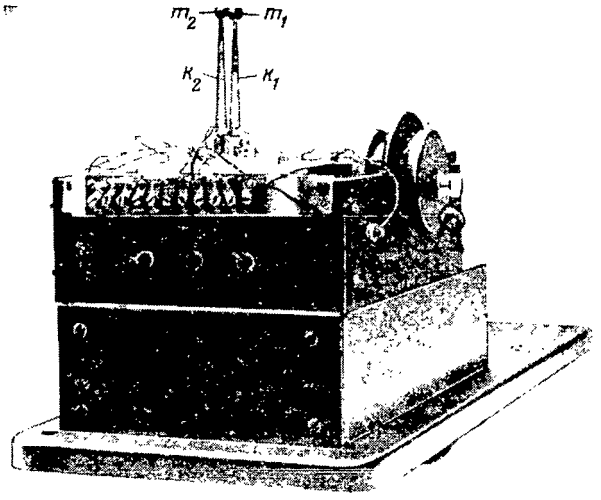

Figure 9.16.

vibration-table panel. It consists of two flat elastic small beams (their rigidities are k_1 and k_2), carrying two small weights m_1 and m_2. The beams are fixed in the body of the model in such a way that the distance between them may be regulated, owing to which, it becomes possible to vary the value of the initial tightness or clearance. In the process of the experiments, the body of the model is imparted a reciprocating vibration. The motions of masses m_1 and m_2 were recorded by means of the pickups of wire resitances glued onto the beams. Moreover, the speed of the vibration-table panel's vibration was also recorded.

Both beams were made of the same material (spring steel) and were of exactly the same size and shape. Consequently, for the model, $\kappa = k_1/k_2 = 1$. Bearing balls were used as the weights m_1 and m_2. The sizes of the balls were /359 $d_1 = 10$ mm, $d_2 = 8$ mm respectively. Thus, the value of μ for the model was about equal to two.

Consequently, the values of parameters κ and μ of the model coincide sufficiently well with those values which were adopted in the construction of characteristics $\lambda = \lambda(\zeta_2)$ and $\delta_s = \delta_s(\zeta_2)$.

The theoretical ratio between the frequencies of the free oscillations of individual parts of the system in this case must be equal to

$$\frac{\omega_2}{\omega_1} \approx 1.41.$$

In Figures 9.17 a, and 9.17 b, we give the oscillograms of the free oscillations of individual parts of system m_1 (Figure 9.17,a) and m_2 (Figure 9.17,b). Their frequencies were found to be respectively equal to $\omega_1 = 88.51$/sec, $\omega_2 = 120.1$/sec. Here $\omega_2/\omega_1 = 1.35$. The insignificant discrepancy between the calculated and actual values of ratio ω_2/ω_1 may be explained evidently by the unequal effect of the pickups glued on them and the unequal conditions of clamping of the beams.

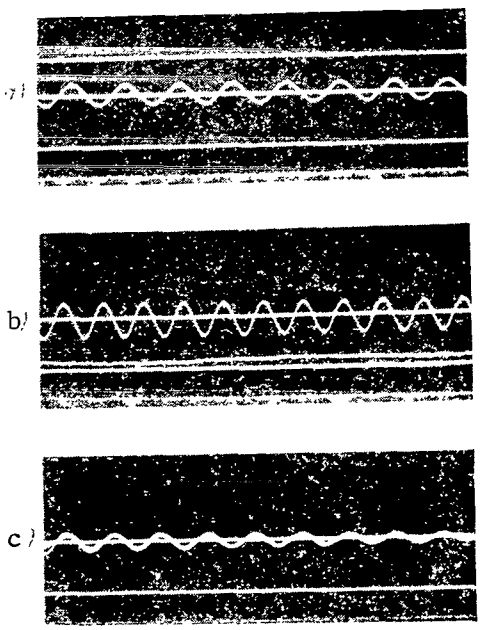

Figure 9.17.

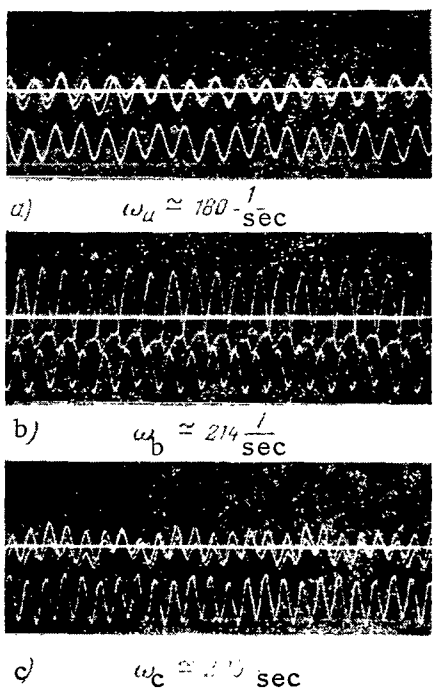

Figure 9.18.

According to calculations, frequencies ω_0 and $\bar{\omega}$ of the free oscillations and with collisions respectively, must be equal to $\bar{\omega} = 216$ 1/sec. $\omega_0 = \sqrt{(1+\kappa)(1+\mu)}\ \omega_2 = 98$ 1/sec, the value of $\bar{\omega}$ calculated for the tightness value of L = 0.2 being indicated here. The measured oscillation frequency was $\omega_0 = 101$ 1/sec. (see Figure 4.17,c).

In Figure 9.18 we give the first series of experiments, performed with this model when L = 0.2. The straight line in each of the pictures corresponds to the position of the static equilibrium of the system. The wavy line in the lower part of the picture portrays the rate of vibration. The actuation frequencies indicated in the oscillograms in Figure 9.18,a, b and c, correspond to the pre-resonance, resonance, and post-resonance conditions of motion. It is not hard to see that in every case the motion of the system is performed with the frequency equal to the actuation frequency. The resonance conditions take /360 place during the actuation frequency of $\omega = 216$ 1/sec, approximately equal to the calculated value of $\bar{\omega}$. Under these actuation conditions the disalignment of both parts of the system increases sharply, which agrees well with the calculation results. Moreover, the forced shifts of both parts of the system differ with respect to sign and are about equal in size, as it should be expected, taking into account relationship (9.74) when $\kappa = 1$. We should also turn our attention to the fact that with equal values of forced shifts, the disalignments of both parts of the system differ sharply, which also agrees well with the calculation results.

The oscillograms in Figure 9.19 are taken during a smooth variation of the actuation frequency.

In the pictures we can see clearly the/361

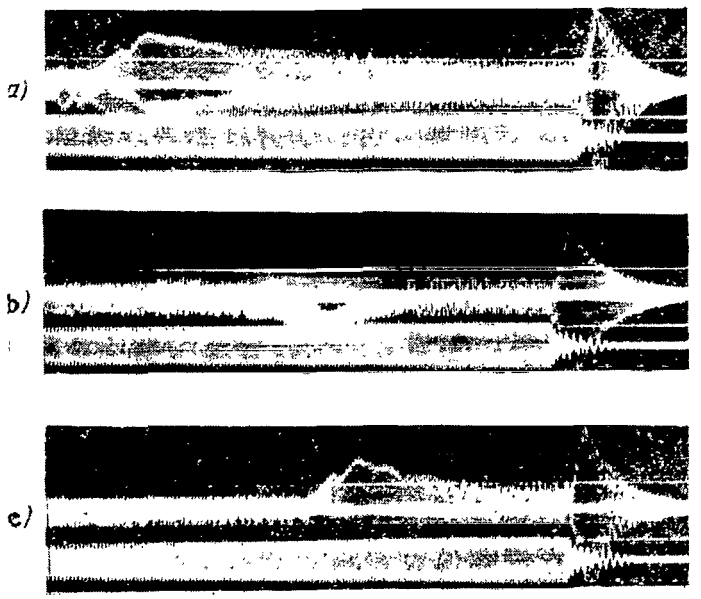

Figure 9.19.

conditions of passage through the resonance, accompanied by a substantial increase of the forced shift and disalignment of the individual parts of the system. In Figure 9.19 a, we recorded the movement of both masses, in Figure 9.19 b, only of mass m_1, and in Figure 9.19 c, only of mass m_2.

In the series of experiments described, the value of the relative tightness was equal to L = 0.2 (1 = 0.2 mm, ρ_0 = 1 mm). We get an idea about how the picture of the system's motion changes upon the change of the initial set-up of the system, from two series of experiments, the results of which are given in Figures 9.20 and 9.21. /362

The oscillograms in Figure 9.20 are taken with the value of the relative tightness of L = 0.5. According to the calculations, with L = 0.5, value $\bar{\omega}$ is equal to $\bar{\omega}$ = 247 1/sec. It is not hard to see that with the increase of tightness, the dynamic errors of the system decrease substantially (compare 9.18,b and 9.20,c) and that the resonance frequency here, shifts towards the increase. These results also agree well with the conclusions which proceed from the theoretical analysis. With an increase of the tightness to L = 0.7 the breaks of the system had a random character and it was not possible to establish the periodic conditions of its motion either.

Finally, in Figure 9.21 we give the oscillograms taken from the model installed with an initial clearance between masses m_1 and m_2 equal to L = -0.2. /363 Such an installation of the system, as it should have been expected, resulted in a growth of the dynamic errors and a certain decrease of the resonance frequency, in accordance with the decrease of frequency $\bar{\omega}$.

In this way, the experiments performed, supported the initial assumption of the presence of stable periodic conditions of motion of asymmetrical two-mass vibro-impact systems, and also verified the calculated character of the course of relationships $\lambda = \lambda(\zeta)$ and $\delta_s = \delta_s(\zeta)$, peculiar to the system examined, with the unlinear (with breaks and collisions) conditions of its work. /364

The results obtained in the last two paragraphs of this chapter supplement

substantially the picture of the influence of factors connected with their real structural execution, on the dynamic accuracy of the mechanisms and systems with elastic couplings.

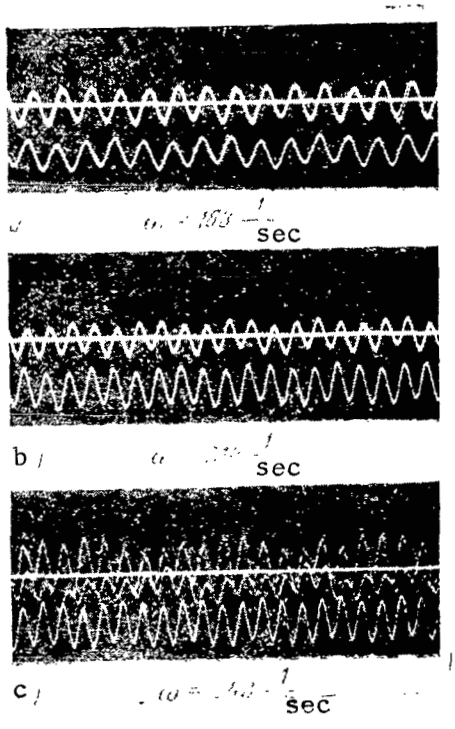

Figure 9.20

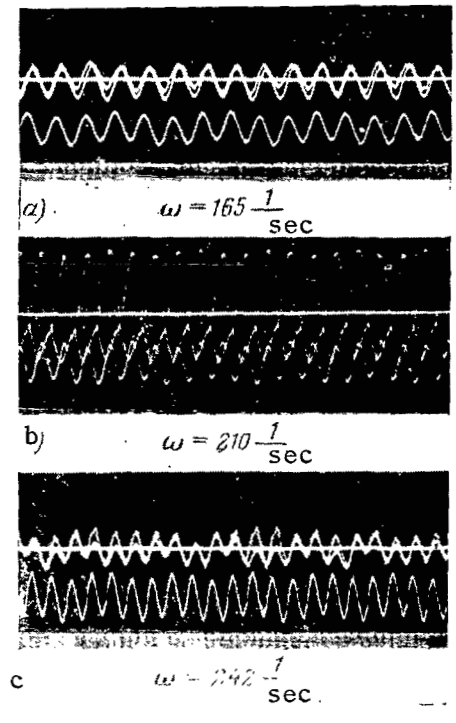

Figure 9.21

Along with the disbalance and friction, the effect of which has been determined in the fifth and sixth chapters, we can now evaluate the dynamic errors arising as the result of collision of elements in kinematic pairs, coupled with a clearance or made with a forced coupling. /365

We should note that the comparative evaluation of the regularities of the course of frequency characteristics of dynamic errors often enables us to judge about the reasons that caused their appearance.

Thus, in particular, in Chapter 5 it was shown that the forced shift arising as the result of disbalance of the mobile part of the system increases monotonously with the increase of the frequency of actuations, and attains an appreciable value only with frequencies, which surpass considerably the frequency of free oscillations of the system. As for the forced shift, occurring as the result of collisions, as it has been shown in this chapter, its frequency characteristic has a sharply expressed resonance character and attains the maximum value with the oscillation frequency located in the frequency range, peculiar to the vibro-impact system.

Keeping this in mind, we can, in the process of vibration tests of the instrument or system, establish according to their behavior during a smooth variation of the vibration frequency, which one of the factors, the disbalance or collisions, is the main cause of the origination of the dynamic errors.

In conclusion, we should state, that the analysis of the periodic vibro-impact conditions of a specific type does not exhaust completely the problems which may arise in the investigation of the dynamics or dynamic precisions of machines, mechanisms, and systems with elastic couplings. At the same time, the qualitative deductions and quantitative relationships, obtained in the last three chapters of this book, may be used as the basis for solving the problems of dynamic synthesis, connected with the selection of the optimum parameters of vibro-impact systems.

APPENDIX

§ 1. The Biharmonic Vibration Table*

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The biharmonic vibration table used for performing these experiments possesses two specific features distinguishing it from other designs of modern vibration tables (see, for example, [33]).

The kinematic system of the vibration table's drive, which includes a special differential mechanism, assures the possibility of smooth regulation, in the process of work, of the frequency and amplitude of the vibration, independently of one another. The motion of the table proceeds not according to the harmonic law but is the result of superposition of two harmonics of different frequency.

The smooth regulation of the frequency and amplitude of the vibration enables us to establish the desired actuation conditions in the most simple way. The biharmonic character of this actuation enables us to obtain the conditions of the mechanism's work approaching the real ones. Actually, in the Introduction it was mentioned that usually a vibration represents a complex polyharmonic process. At the same time, all the theoretical and experimental investigations, as well as technical conditions for acceptance of individual mechanism, instruments, and machine units, are performed and formulated for the case of monoharmonic actuation.

Questions may arise about how do the results obtained here change, if the harmonic process of actuation will be modulated with other harmonics, will the periodic character of the motion of two-mass systems, examined in Chapters 8 and 9 for example, be preserved in this case, will the periodic conditions of motion be stable under the conditions of such actuation, etc.

The theoretical analysis of these problems is extremely complicated. At /367
the same time, using biharmonic actuation, we can experimentally evaluate the effect of the action of the components, modulating the main harmonic actuation, and verify in this way the acceptability of the results obtained in the assumption of the harmonic character of the excitation.

In Figure 1 we portray the system of actuation of vibration, used in the instrument described. Mass M is a mass of mobile parts of the instrument, with the deduction of the masses of six little weights, located symmetrically with respect to the longitudinal axis of the vibrator. The entire instrument, just as the weights m, moves in the horizontal plane and is centered with springs which have very low rigidity.

Two of these weights, located on one side of the axis of symmetry of the

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* The author's certificate No. 78109 was issued to A.Ye. Kobrinskiy in 1949 (see also [36]).

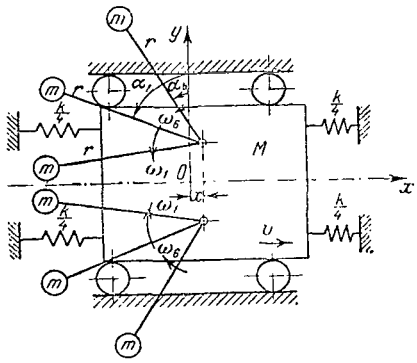

Figure 1.

of the table, rotate with a similar angular velocity ω_1, the third weight moves with the angular velocity ω_6. Angle ϕ between the weights may be changed in the process of motion of the table, by means of a device described below.

According to the designations adopted in Figure 1, the equation of motion of the vibration table may be written as follows:

$$(M+6m)\ddot{\varrho}+k\varrho=-2mr\omega_1^2[\sin\omega_1 t+\sin(\omega_1 t+\varphi)]-2mr\omega_6^2\sin\omega_6 t.$$

Consequently, the established motion of the table will be

$$\varrho=\frac{2mr\omega_1^2}{M+6m}\cdot\frac{1}{\omega^2-\omega_1^2}[\sin\omega_1 t+\sin(\omega_1 t+\varphi)]+\frac{2mr\omega_6^2}{M+6m}\cdot\frac{1}{\omega^2-\omega_6^2}\sin\omega_6 t.$$

Here ω is the frequency of the free oscillations of the instrument's system. /368
This frequency is so small that with a very small error, we may disregard its effect on the amplitude of the vibration. Then the law of motion of the vibration table assumes the following appearance:

$$\varrho\approx\frac{4mr\cos\frac{\varphi}{2}}{M+6m}\cdot\sin\left(\omega_1 t+\frac{\varphi}{2}\right)+\frac{2mr}{M+6m}\sin\omega_6 t$$

and represents the sum of two harmonics, having different frequency and amplitude. According to this expression, the amplitude of the second harmonic (ω_6) remains constant, and the amplitude of the first harmonic depends on the size of the angle between the weights, and by the variation of this angle it can be regulated within the limits from zero to the value 4mr/M + 6m.

In Figure 2 we give the kinematic diagram of the vibrator, while the design of its mechanism is given in Figure 3. According to the kinematic diagram, the driving roller of the instrument is rigidly coupled with gears 1 and 10. On the roller, gears 12 and 8 are set freely, as well as gear 6 representing the carrier of satellite gears 2 and 4. Gears 6, 8 and 10 carry the unbalanced weights and are engaged with gears 7, 9 and 11, set up on the loose-running roller and equipped with weights suitably arranged. Satellite gears 2 and 4 run around gears 3 and 5, one of which is rigidly coupled with the body of the instrument. Gear 3 is fastened so that it has mobility and /370
is equipped with a handle, by means of which it may be either fixed in a specific position or turned, changing the setting angle between gears 8, 10 and 9 and 11 respectively.

When the handle does not move (ω_3 = 0) gears 8 and 10 rotate in the same

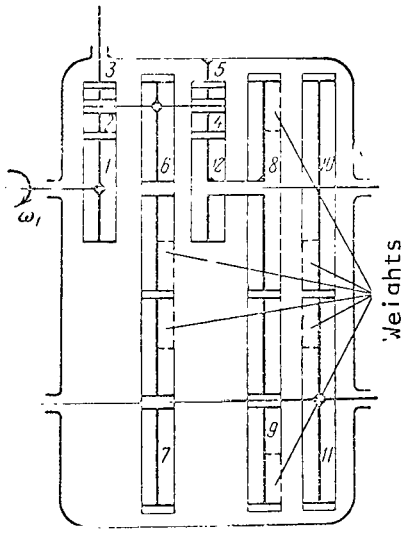

Figure 2.

direction, with the same angular velocity, designated with ω_1. Here, the angular velocity of gear 6 is determined from the gear ratio

$$i_{1-6} = \frac{z_1 + z_3}{z_1}.$$

In the manufactured structure $z_1 = 20$, $z_3 = 52$, and therefore

$$\omega_6 = \frac{5\omega_1}{18}.$$

If in the process of motion of the mechanism we impart an additional motion to wheel 3, then the paired gears 12 and 8 get an additional angle of turn with respect to gear 10, connected with the angle of turn of the handle of the ratio

$$\varphi_8 = \frac{z_3}{z_1}\varphi_3 = \frac{13}{5}\varphi_3.$$

/369

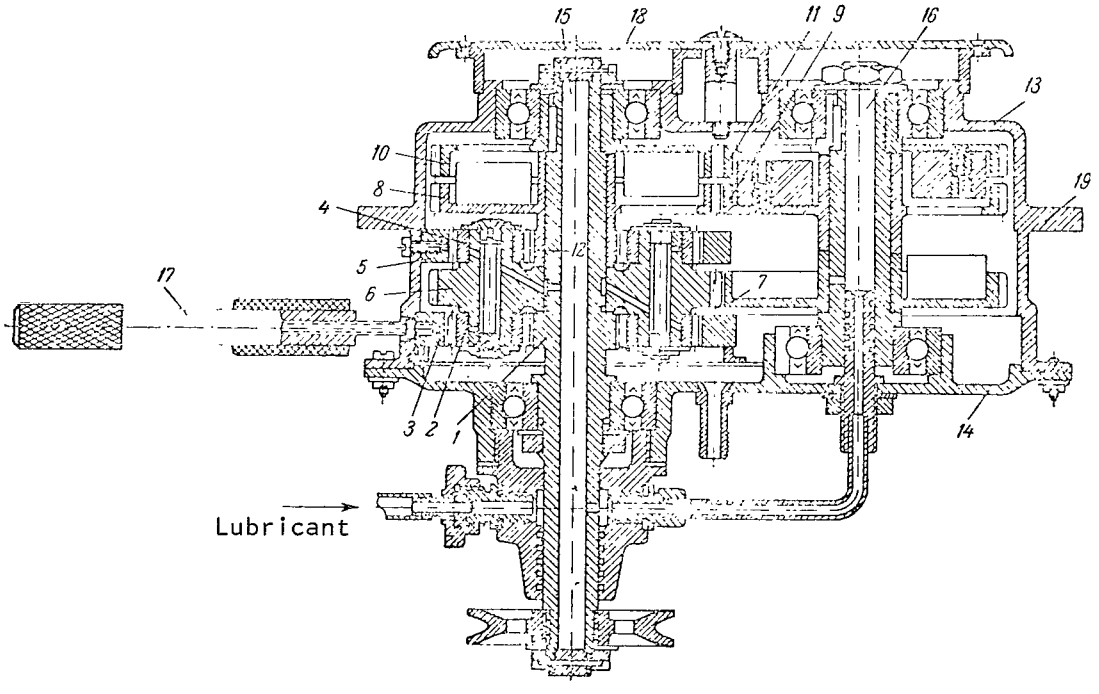

Figure 3.

The design of the vibration table is presented in Figures 3 and 4. The vibration mechanism of the table (see Figure 3) is enclosed in body 13, made of a light alloy. The lower part of the body is covered with a lid 14, centered on the body. In the body the driving roller 15 and free-running roller 16 are installed on ball bearings, and gear 5 with inside teeth is pressed in. Gear 3, similar to gear 5 is set freely on the body, while its fractional movement is limited by a spring ring. Handle 17, screwed into gear 3, enables us to turn it in the process of the work of the instrument, regulating the amplitude of the oscillations.

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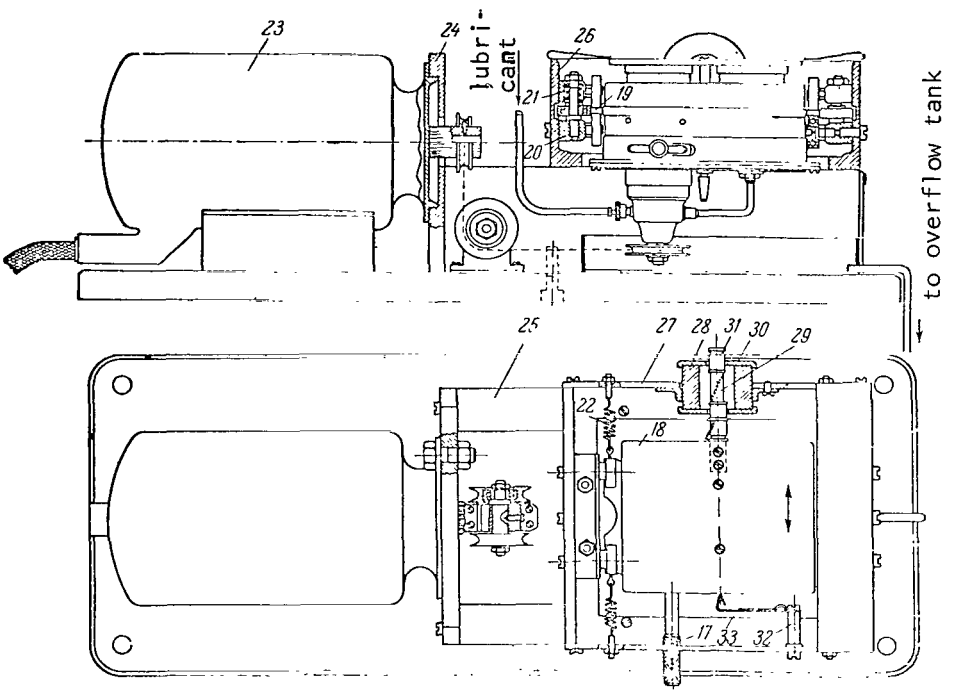

Figure 4.

On roller 15 gear 1 is cut and gear 10 is rigidly fixed, and also gear 6 is set, which serves as the carrier of satellite gears 2 and 4, and the paired gears 8 and 12. Gear 11 is fixed and gears 7 and 9 are set up freely on roller 16.

On every one of gears 6, 7, 8, 9, 10 and 11 a lead weight is attached; in order to increase the disbalance, in the wall of the gear at a 180° angle to the weight, a hole is drilled, the shape of which coincides with the shape of the weight. /372

The disbalance of each gear was established at

$$m = 0.045 \text{ g} \cdot \text{sec}^2/\text{cm}.$$

The center of gravity of mass m, representing a part of the ring is located at the distance

$$r_s = \frac{2}{3}\frac{R^3 - r^3}{R^2 - r^2}\frac{\sin\beta}{\beta} = 1{,}9 \text{ см}.$$

from the axis of rotation.

The mass of the mobile parts of the vibrator (deducting the mass of the six weight) was determined by weighing and found to be

$$M = 1.79 \text{ g} \cdot \text{sec}^2/\text{cm}.$$

For the indicated values of M, m, and r_s the law of motion of the vibrator will be written as follows:

$$\varrho = 0{,}165 \cos\frac{\varphi}{2} \sin\left(\omega_1 t + \frac{\varphi}{2}\right) + 0{,}083 \sin\omega_6 t, \text{ см}.$$

On the upper part of the body of the vibrator the vibrator table is installed, on which the objects tested are fixed. Four ledges 19 are made on the body, which serve for directing the movement of the vibrator in its work (see Figure 4). The position of the vibrator with respect to the directing cross-pieces is fixed by four springs 22, the rigidity of which does not exceed k = 0.3 kg/cm.

When tested with four springs and an elastic beam of the displacement pickup (see below), the free frequency of the vibrator did not exceed 25 1/sec. This value may be decreased taking off two of the four springs.

The vibrator was activated by the electric motor 23, through a flexible roller. A 1 kw DC electric motor with 6,000 rpm guarantees the feasibility of smooth regulation of vibration frequency.

The design provides for the possibility of simultaneous and independent recording of displacement and speed of the table.

The displacement is recorded by means of a wire resistance pickup. The displacement pickup is made according to the system described in [60] and consists of an elastic beam 33 (see Figure 4), fixed onto column 32, connected /373 with the beam of the instrument. In the process of displacement of the table the beam is deformed, as the result of which the resistance of the pickup glued onto the beam, changes. At the same time the power of the current passing through the pickup changes. This current is increased by means of special amplifiers, and then arrives at the loop of the oscillograph.

Speed is recorded by a magnetoelectric pickup. This pickup is made in the following manner. On channel-bar 26 (Figure 4) plate 27 is fixed, carrying a permanent magnet 28 of cylindrical shape equipped with pole tips 29, made of soft iron. On the rod two coils are wound, which move with respect to the

different poles. Upon the movement of the table the loops of the coil cross the lines of force of the magnetic field, and in them an electromotive force is generated, proportional to the speed of motion of the table. The signal from the speed pickup, just as the signal from the displacement pickup, arrives/374 at the oscillograph's loop.

In Figure 5 we show the photographs of the table and in Figure 6, the typical oscillograms of displacement and speed of the table of the vibrator. The actuation period is $T = 5T_6 = 18T_1$.

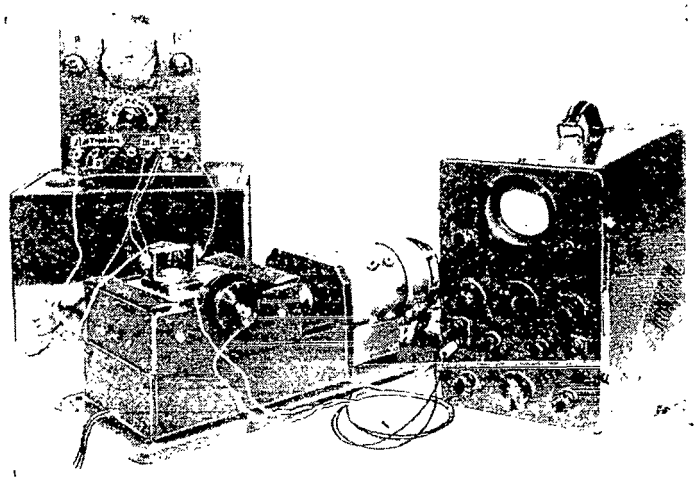

Figure 5.

In the displacement oscillogram the lower frequency component (ω_6), the modulated component of high frequency (ω_1), is sharply expressed. On the speed oscillogram, the picture changes. The value of the speed is mainly determined by the high-frequency harmonic, while the high-frequency component is slightly modulated by the low-frequency component. Changing the position of the handle, we may, without stopping the vibration table, determine the most diverse laws of actuation.

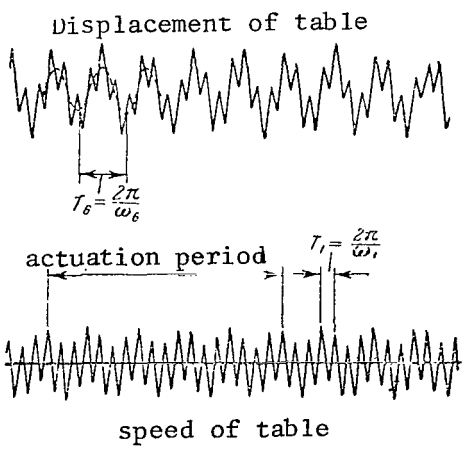

Figure 6.

§ 2. Calculation of a Hinged Four-Component System.

This appendix was written with the purpose of: 1) giving a visual illustration of the determination of various values, used with the dynamic analysis of mechanisms with elastic couplings, and 2) to establish the relationships required in the calculation of an experimental four-component system (see § 5.8).

The mechanism examined is shown in /375 Figure 7. The elastic couplings included in the mechanism, represent a helicoidal spring ED and a flat spiral spring GH. The elastic characteristics of these couplings are assumed to be linear. The rigidity of the helicoidal spring is equal to k, the torsion rigidity of the spiral spring is equal to q.

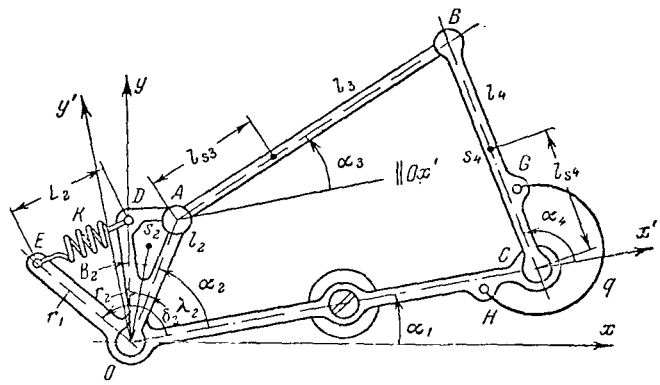

Figure 7.

The mechanism is in a vertical plane, its location in this plane being characterized by angle α_1. In addition to the elastic forces of the couplings, the mechanism is acted upon by the force of the weights of its components. For generalizations of the coordinate we select angle $\alpha = \alpha_2$.

a) Equilibrium of the Mechanism. The position of the mechanism with elastic couplings is determined from the equation

$$\frac{d\Pi}{d\alpha_2} = \frac{d\Pi_{\text{I}}}{d\alpha_2} + \frac{d\Pi_{\text{II}}}{d\alpha_2} = 0.$$

The potential energy of the forces of the weight of the components is equal to

$$\Pi_{\text{I}} = G_2 l_{s2} \sin(\alpha_1 + \alpha_2 + \lambda_2) + G_3 [l_2 \sin(\alpha_1 + \alpha_2) + l_{s3} \sin(\alpha_1 + \alpha_3)] + G_4 [l_1 \sin\alpha_1 + l_{s4} \sin(\alpha_1 + \alpha_4)] + \Pi_0. \quad (1)$$

In this equation, G_2, G_3, G_4 are weights of the components of the mechanism. In determining the potential energy of deformation of the elastic couplings we use the following designations: α_{22} is the value of the generalized coordinate with which the elastic deformation of coupling ED is equal to zero, and α_{24} is the same for coupling GH, δ_2 and δ_4 are elastic deformations of couplings ED and GH. The potential energy of deformation of the elastic couplings is equal to /376

$$\Pi_{\text{II}} = k\delta_2 + q\delta_4$$

and, consequently

$$\frac{d\Pi}{d\alpha_2} = \frac{d\Pi_{\text{I}}}{d\alpha_2} + k\delta_2 \frac{d\delta_2}{d\alpha_2} + q\delta_4 \frac{d\delta_4}{d\alpha_2}.$$

The current value of δ_2 will be determined from triangle OED (see Figure 7):

$$\delta = L_2 - L_{22},$$

where

$$L_2^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\alpha_2 + \beta_2 - \gamma_2),$$

from which

$$\delta_2 = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\alpha_2 + \beta_2 - \gamma_2)} - L_{22}. \quad (2)$$

Angle α_{22} with a known initial dimension L_{22} of coupling ED may be found, specifying that $\delta = 0$, from the relationship

$$\cos(\alpha_{22} + \beta_2 - \gamma_2) = \frac{r_1^2 + r_2^2 - L_{22}^2}{2r_1 r_2}. \quad (3)$$

We determine the current value of δ_4 by projecting the closed vectorial contour OABC (see [30]) on the coordinate axes x' and y':

$$-l_1 + l_2 \cos\alpha_2 + l_3 \cos\alpha_3 = l_4 \cos\alpha_4, \quad (4)$$
$$l_2 \sin\alpha_2 + l_3 \sin\alpha_3 = l_4 \sin\alpha_4. \quad (5)$$

We designate

$$l_2 \cos\alpha_2 - l_1 = a, \quad l_2 \sin\alpha_2 = b.$$

Then, equations (4) and (5) assume the following appearance:

$$a + l_3 \cos\alpha_3 = l_4 \cos\alpha_4, \quad (6)$$
$$b + l_3 \sin\alpha_3 = l_4 \sin\alpha_4, \quad (7)$$

from which

$$\alpha_4 = \arccos \frac{a + l_3 \cos\alpha_3}{l_4}.$$

For determining angle α_3 we will square equations (6) and (7) and add them up. As the result we shall get a quadratic equation /377

$$(1 + B^2)\cos^2\alpha_3 + 2A\cos\alpha_3 + (A^2 - B^2) = 0,$$

where

$$A = \frac{a^2 + b^2 + l_3^2 - l_4^2}{2al_3}, \quad B = \frac{b}{a}.$$

Now we can write the value of deformation δ_4 as follows:

$$\delta_4 = \alpha_4 - \alpha_{24} = \arccos \frac{a + l_3 \cos \alpha_3}{l_4} - \alpha_{24}. \tag{8}$$

Angle α_{24} corresponding to the undeformed coupling GH ($\delta_4 = 0$), can be found also by using equations (4) and (5).

Now we turn to setting up the equations of equilibrium, for which we form the expressions for derivatives $\frac{d\delta_2}{d\alpha_2}$ and $\frac{d\delta_4}{d\alpha_2}$.

Differentiating (2) we get

$$\frac{d\delta_2}{d\alpha_2} = \frac{r_1 r_2 \sin(\alpha_2 + \beta_2 - \gamma_2)}{[r_1^2 + r_2^2 - 2r_1 r_2 \cos(\alpha_2 + \beta_2 - \gamma_2)]^{1/2}}. \tag{9}$$

Differentiating (4) and (5) we get

$$l_2 \sin \alpha_2 + l_3 \left(\frac{d\alpha_3}{d\alpha_2}\right) \sin \alpha_3 = l_4 \left(\frac{d\alpha_4}{d\alpha_2}\right) \sin \alpha_4, \tag{10}$$

$$l_2 \cos \alpha_2 + l_3 \left(\frac{d\alpha_3}{d\alpha_2}\right) \cos \alpha_3 = l_4 \left(\frac{d\alpha_4}{d\alpha_2}\right) \cos \alpha_4. \tag{11}$$

Projecting contour OABC onto the axis of components AB we find

$$l_2 \sin(\alpha_2 - \alpha_3) = l_4 \left(\frac{d\alpha_4}{d\alpha_2}\right) \sin(\alpha_4 - \alpha_3),$$

from which

$$\frac{d\alpha_4}{d\alpha_2} = \frac{d\delta_4}{d\alpha_2} = \frac{l_2}{l_4} \frac{\sin(\alpha_2 - \alpha_3)}{\sin(\alpha_4 - \alpha_3)}. \tag{12}$$

Projecting contour OABC on the axis of component BC we get

$$\frac{d\alpha_3}{d\alpha_2} = \frac{l_2}{l_3} \frac{\sin(\alpha_2 - \alpha_4)}{\sin(\alpha_4 - \alpha_3)}. \tag{13}$$

Now, differentiating with respect to α_2 the equation (1) and having in view that

$$\frac{d\Pi_{II}}{d\alpha_2} = k \frac{d\delta_2}{d\alpha_2} + q \frac{d\delta_4}{d\alpha_2}, \tag{14}$$

we can set up the equation of the equilibrium of a hinged four-component system. Value $\alpha = \alpha_{20}$, determined from this equation characterizes the position of the /378 stable equilibrium of the mechanism.

b) Coefficients of the Mechanism. Knowing the position of the equilibrium of the mechanism we can proceed to the calculation of the inertial and quasi-elastic coefficients.

For the hinged four-component system examined the inertial coefficient is equal to

$$J = J_{2s} + J_{3s}\left(\frac{d\alpha_3}{d\alpha_2}\right)_0^2 + J_{4s}\left(\frac{d\alpha_4}{d\alpha_2}\right)_0^2 + m_2 r_{s2}^2 + m_3\left(\frac{ds_3}{d\alpha_2}\right)_0^2 + m_4 r_{s4}^2\left(\frac{d\alpha_4}{d\alpha_2}\right)_0^2, \qquad (15)$$

where m_2, m_3, m_4, J_{2s}, J_{3s}, J_{4s} are respectively the masses of the mechanism's components, and their moments of inertia with respect to the centers of gravity.

Derivatives $\frac{d\alpha_3}{d\alpha_2}$ and $\frac{d\alpha_4}{d\alpha_2}$ contained in (15) were formed earlier by us (see (12) and (13)) and for determining coefficient J it remains to calculate $\frac{ds_3}{d\alpha_2}$. For this we resort to the examination of the turned plan of the mechanism's speeds, presented in Figure 8.

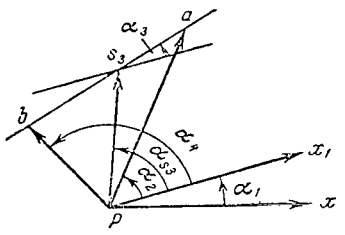

Figure 8.

From triangle ps_3a we get

$$(ps_3)^2 = (pa)^2 + (as_3)^2 - 2(pa)(as_3)\cos(\alpha_2 - \alpha_3).$$

Substituting the values of the segments and reducing the left and right parts of the equation by $\mu_v \dot{\varepsilon}$ we get

$$\left(\frac{ds_3}{d\alpha_2}\right)_0^2 = l_2^2 + l_{s3}^2\left(\frac{d\alpha_3}{d\alpha_2}\right)_0^2 - 2l_2 l_{s3}\left(\frac{d\alpha_3}{d\alpha_2}\right)\cos(\alpha_2 - \alpha_3). \qquad (16)$$

At the same time we determine angle α_{s3}. Using the sinus theorem we find

$$\sin(\alpha_{s3} - \alpha_2) = \frac{l_{s3}\left(\frac{d\alpha_3}{d\alpha_2}\right)}{\left(\frac{ds_3}{d\alpha_2}\right)_0}\sin(\alpha_2 - \alpha_3). \qquad (17)$$

We turn to the determination of the quasi-elastic coefficient of the mechanism. /379

Differentiating (9) and substituting value $\alpha_2 = \alpha_{20}$, we get

$$\frac{d^2\delta_2}{da_2^2} = \frac{r_1 r_2 \cos(\alpha_{20}+\beta_2-\gamma_2)}{[r_1^2+r_2^2-2r_1r_2\cos(\alpha_{20}+\beta_2-\gamma_2)]^{1/2}} + \frac{r_1^2 r_2^2 \sin^2(\alpha_{20}+\beta_2-\gamma_2)}{[r_1^2+r_2^2-2r_1r_2\cos(\alpha_{20}+\beta_2-\gamma_2)]^{3/2}}. \quad (18)$$

Differentiating (10) we find

$$l_2\cos\alpha_2 + l_3\left(\frac{d\alpha_3}{da_2}\right)^2\cos\alpha_3 + l_3\left(\frac{d^2\alpha_3}{da_2^2}\right)\sin\alpha_3 = l_4\left(\frac{d\alpha_4}{da_2}\right)^2\cos\alpha_4 + l_4\left(\frac{d^2\alpha_4}{da_2^2}\right)\sin\alpha_4.$$

Subtracting angle α_3 from this equation and substituting value $\alpha_2 = \alpha_{20}$, we get

$$\left(\frac{d^2\alpha_4}{da_2^2}\right)_0 = \left(\frac{d^2\delta_4}{da_2^2}\right)_0 = \frac{l_2\cos(\alpha_{20}-\alpha_3)+l_3\left(\frac{d\alpha_3}{da_2}\right)_0 - l_4\left(\frac{d\alpha_4}{da_2}\right)^2\cos(\alpha_4-\alpha_3)}{l_4\sin(\alpha_4-\alpha_3)}. \quad (19)$$

Finally, differentiating (1) twice, we get the value:

$$\left(\frac{d^2\Pi_1}{da_2^2}\right)_0 = -\Big\{G_2 l_{s2}\sin(\alpha_1+\alpha_{20}+\lambda_2) + G_3\left[l_2\sin(\alpha_1+\alpha_{20}) + l_{s3}\sin(\alpha_1+\alpha_3)\left(\frac{d\alpha_3}{da_2}\right)_0 - l_{s3}\cos(\alpha_1+\alpha_3)\left(\frac{d^2\alpha_3}{da_2^2}\right)_0\right] + G_4 l_{s4}\left[\sin(\alpha_1+\alpha_4)\left(\frac{d\alpha_4}{da_0}\right)_0^2 - \cos(\alpha_1+\alpha_4)\left(\frac{d^2\alpha_4}{da_2^2}\right)_0\right]\Big\}. \quad (20)$$

Now, keeping in mind that

$$\left(\frac{d^2\Pi}{da_2^2}\right) = \left(\frac{d^2\Pi_1}{da_2^2}\right) + k\left[\left(\frac{d\delta_2}{da_2}\right)^2 + \delta_2\left(\frac{d^2\delta_2}{da_2}\right)\right] + q\left[\left(\frac{d\delta_4}{da_2}\right)^2 + \delta_4\left(\frac{d^2\delta_4}{da_2}\right)\right],$$

and making use of equations (9), (12), (18), (19), and (20), we can calculate /380 the sought value of the quasi-elastic coefficient, equal to

$$K = \left(\frac{d^2\Pi}{da^2}\right)_0.$$

Determining value α_{20} for various positions of the mechanism with respect to axis x (i.e., for various α_1) and calculating the corresponding values of coefficients J and K, we can plot a statical characteristic of the mechanism and a characteristic of the frequency of free oscillations.

Now we shall prescribe to ourselves a definite law of actuation and begin

the determination of the coefficients of periodicity and excitation.

We assume that the stand of the mechanism performs reciprocating vibrations according to the harmonic law in the direction of the y axis. If the position of the dynamic equilibrium, establishing itself under the conditions of the stand's vibration, differs little from the position of static equilibrium (i.e., if the forced shift of the mechanism is small), then the coefficient of periodicity in the general instance is approximately determined as follows:

$$\Psi_2 = \frac{1}{J}\sum_0^n m_i \left(\frac{ds_i}{d\alpha_2}\right)_0 \left(\frac{d\alpha_i}{d\alpha_2}\right)_0 \sin(\psi_0 - \alpha_{si0}).$$

Since the values of derivatives $\frac{ds_i}{d\alpha_2}$ and $\frac{d\alpha_i}{d\alpha_2}$ were determined by us earlier, now the problem consists only of finding the sizes of angles (ψ_0 - α_{si0}). For this we shall construct a turned plan of the relative velocities of the mechanism (Figure 9) and apply at the pole of the plan vector $\bar{u}$ of the transfer velocity of vibration.

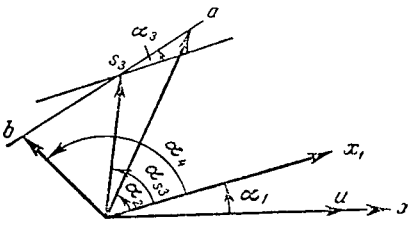

Figure 9.

By means of this velocity plan, the periodicity coefficient is determined as follows: /381

$$\Psi_2 = \frac{1}{J}\left[m_2 l_{s2}\sin(\alpha_{20}+\alpha_1) + m_3\left(\frac{ds_3}{d\alpha_2}\right)_0\left(\frac{d\alpha_3}{d\alpha_2}\right)_0 \sin(\alpha_{s3}+ \right.$$
$$\left. + \alpha_1) + m_4\left(\frac{ds_4}{d\alpha_2}\right)_0\left(\frac{d\alpha_4}{d\alpha_2}\right)_0 \sin(\alpha_1+\alpha_4)\right]. \quad (21)$$

The coefficient of excitation in the case examined is

$$\Theta_2 = \frac{1}{J}\left[m_2 l_{s2}\cos(\alpha_{20}+\alpha_1) + m_3\left(\frac{ds_3}{d\alpha_2}\right)_0 \cos(\alpha_{s3}+\alpha_1) + m_4\left(\frac{ds_4}{d\alpha_2}\right)\sin(\alpha_1+\alpha_4)\right] \quad (22)$$

The results of the calculations, performed according to the previously derived formulas, are given in § (5.8).

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Translated for the National Aeronautics and Space Administration under Contract No. NASw-1695 by Techtran Corporation, P.O. Box 729, Glen Burnie, Maryland, 21061.

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